FORM-FINDING OF SELF-STRESSED STRUCTURES BY AN EXTENDED FORCE DENSITY METHOD

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ABSTRACT

Extended from the basic idea and formulations of force density method initially developed for cable nets, this paper presents a numerical method for the form-finding problem of self-stressed structures. Singular value decomposition of the equilibrium matrix is utilized to find a feasible set of force densities satisfying the requirement on rank deficiency. A unique configuration of the structure can be obtained by specifying an independent set of nodal coordinates. The proposed method can have some but not exact controls over the geometrical and mechanical properties of a structure. Its ability of searching new configurations and good convergence are demonstrated by several numerical examples.

Keywords: self-stressed structure, form-finding, singular value decomposition

1. INTRODUCTION

The self-stressed structure being dealt with in this study consists of two distinct kinds of members: cables, which transmit only tension, and struts, which transmit only compression. No physically touching between any two struts [1]. Furthermore, it is free-standing, where no fixed nodes (supports), and the members are assumed to be pin-jointed by ends.

Self-stressed structure is unstable while in the unstressed state. So, in order to obtain a stable structure, it is necessary to introduce prestress to the structure to stiffen the mechanisms. As geometrical shape and prestress distribution of a self-stressed structure are highly interdependent, its configuration has to satisfy determined taking consideration both of these two factors simultaneously, which is well known as *form-finding*.

To the form-finding problem of self-stressed structures, methods extended from the basic idea and formulations of force density method, which is originally developed for cable nets [2], are thought to be very effective since only linear equations need to be solved. Among these methods, the analytical technique [3], which analyzes the equilibrium matrix in multi-parameter symbolic form to find the necessary condition of force densities for the required rank deficiency, is found to be particularly suitable for finding new configurations with given topology [4]. However, the force densities in this method have to be analyzed in symbolic forms, which is thought to be not effective enough for a relatively complicated structure, e.g. a structure with many members. So the objective of this study is to present a numerical method that finds the feasible set of force densities, which ensures equilibrium matrix has the required rank deficiency, based on singular value decomposition (SVD) of the equilibrium matrix.

Following this introduction, Section 2 formulates the equilibrium equations based on the force density method, gives the linear relation of force density vector and components of equilibrium matrix and the linear constraints on force densities as well, and then presents design procedure based on singular value decomposition (SVD). Some numerical examples are given in Section 3 to illustrate good convergence and ability of searching new configurations of the proposed method. Section 4 concludes and gives some discussions on this study.

2. EXTENDED FORCE DENSITY METHOD

2.1 Equilibrium Analysis

For the formulation of equilibrium equations of a self-stressed structure, we introduce the following assumptions:

(a). topology of the structure is known;

(b). members are connected by pin joints;

(c). no external load is applied and the self-weight is neglected;

(d). buckling of the struts is not considered;

(e). it is free-standing, without any support.

From these assumptions, we may know that

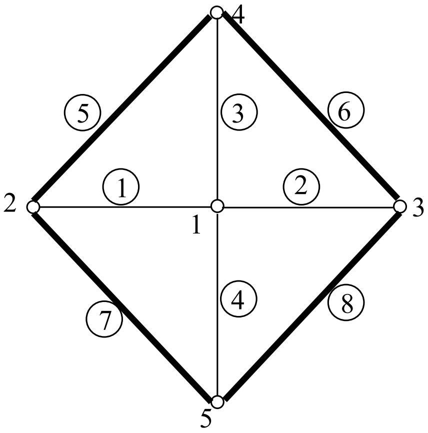


Fig. 1: A two dimensional tensegrity structure.

1. geometrical configuration of the structure can be described in terms of nodal coordinates;
2. members of the structure transmit only axial forces, either in compression or in tension;
3. the structure is in a self-equilibrium state.

Consider a self-stressed structure with m members and n nodes. The force density q_k of member k is defined as its axial member force s_k to length l_k ratio ($q_k = s_k/l_k$), A positive and negative value of the force density q_k indicates that member k is in tension and in compression, respectively. The force density vector $\mathbf{q}$ is denoted as $\mathbf{q}=(q_1, q_2, \ldots, q_m)^\top$. Self-equilibrium equations of the structure in $x-$, $y-$ and $z-$ directions, respectively, can be written as follows [2]

$$\begin{aligned}\mathbf{Ex}&=\mathbf{0}\\ \mathbf{Ey}&=\mathbf{0}\\ \mathbf{Ez}&=\mathbf{0}\end{aligned} \tag{1}$$

where $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}(\in \Re^n)$ are the nodal coordinate vectors in $x-$, $y-$ and $z-$ directions, respectively. Since $\mathbf{E} \in \Re^{n\times n}$ indicates equilibrium of the structure with respect to nodal coordinates, it is called *equilibrium matrix* in this study. It can be calculated by

$$\mathbf{E} = \mathbf{C}^\top \mathbf{Q}\mathbf{C} \tag{2}$$

where $\mathbf{C} \in \Re^{m\times n}$ describes the topology of the structure and the force density matrix $\mathbf{Q}$ is the diagonal matrix of $\mathbf{q}$ ($\mathbf{Q}$=diag($\mathbf{q}$)). In the case of self-stressed structure, $\mathbf{E}$ is always square, symmetry but *singular*, because the structure does not have supports and the values of the force densities may be negative or positive. Therefore, configuration of the structure, which is described in terms of nodal coordinates, cannot be uniquely determined by Eq. (1).

Define the rank deficiency of $\mathbf{E}$ as

$$h = n - \mathrm{rank}(\mathbf{E}) \tag{3}$$

It is obvious that there are h independent nodal coordinates in x-, y- and z-directions, respectively, that can be arbitrarily specified, for satisfying Eq. (1).

Geometrically, four independent points form a three-dimensional space. Accordingly, the required rank deficiency $h^\star$ has to be $h^\star = 4$, so as to achieve a three-dimensional stable structure, which can be neither moved nor rotated.

2.2 Formulation of Force Density Vector

Instead of using $\mathbf{C}$ and $\mathbf{Q}$ as in Eq. (2), the equilibrium matrix $\mathbf{E}$ can be written directly from the force densities [3]. Let $\mathcal{I}$ denote the set of members connected to free node i. The (i,j) component $\mathbf{E}_{(i,j)}$ of $\mathbf{E}$ is given as

$$\mathbf{E}_{(i,j)} = \begin{cases} \sum\limits_{k\in\mathcal{I}} q_k & \text{for } i = j \\ -q_k & \text{if free nodes } i \text{ and } j \text{ are connected by member } k \\ 0 & \text{for other cases} \end{cases} \tag{4}$$

For a two-dimensional self-stressed structure as shown in Fig. 1, where the thick and thin lines denote struts and cables, respectively, $\mathbf{E}$ can be written as follows by using Eq. (4):

$$\mathbf{E} = \begin{pmatrix} q_1+q_2+q_3+q_4 & -q_1 & -q_2 & -q_3 & -q_4 \\ -q_1 & q_1+q_5+q_7 & 0 & -q_5 & -q_7 \\ -q_2 & 0 & q_2+q_6+q_8 & -q_6 & -q_8 \\ -q_3 & -q_5 & -q_6 & q_3+q_5+q_6 & 0 \\ -q_4 & -q_7 & -q_8 & 0 & q_4+q_7+q_8 \end{pmatrix}$$

The ith column $\mathbf{E}_i$ of $\mathbf{E}$ can be written in terms of force density vector $\mathbf{q}$ by a matrix $\mathbf{B}^i \in \Re^{n\times m}$ as

$$\mathbf{B}^i\mathbf{q} = \mathbf{E}_i \tag{5}$$

where (j, k) $(k \in \{1, 2, \ldots, m\})$ component $\mathbf{B}^i_{(j,k)}$ of $\mathbf{B}^i$ is defined as

$$\mathbf{B}^i_{(j,k)} = \begin{cases} 1 & \text{if } i = j \text{ and } k \in \mathcal{I} \\ -1 & \text{if nodes } i \text{ and } j \text{ are connected by member } k \\ 0 & \text{for other cases} \end{cases} \tag{6}$$

By letting $\mathbf{B}^\top = (\mathbf{B}^{1\top}, \ldots, \mathbf{B}^{j\top}, \ldots, \mathbf{B}^{n\top})$ and $\mathbf{g}^\top = (\mathbf{E}_1^\top, \ldots, \mathbf{E}_i^\top, \ldots, \mathbf{E}_n^\top)$, force density vector $\mathbf{q}$ can be expressed in terms of components of $\mathbf{E}$ as

$$\mathbf{B}\mathbf{q} = \mathbf{g} \tag{7}$$

Linear constraints on some specific force densities, such as having expected values, can be formulated as

$$\mathbf{B}^e\mathbf{q} = \mathbf{q}^e \tag{8}$$

where $\mathbf{B}^e$ and $\mathbf{q}^e$ denote the constraint matrix and the vector containing expected values, respectively. By letting $\bar{\mathbf{B}}^\top = (\mathbf{B}^\top, \mathbf{B}^{e\top})$ and $\bar{\mathbf{g}}^\top = (\mathbf{g}^\top, \mathbf{q}^{e\top})$, we obtain

$$\bar{\mathbf{B}}\mathbf{q} = \bar{\mathbf{g}} \tag{9}$$

Since matrix $\bar{\mathbf{B}}$ is full-rank, known by its definition for a self-stressed structure, the least square solution of force density vector $\mathbf{q}$ can be calculated by [5]

$$\mathbf{q} = \bar{\mathbf{B}}^-\bar{\mathbf{g}} \tag{10}$$

where $\bar{\mathbf{B}}^-$ denotes the generalized inverse matrix of $\bar{\mathbf{B}}$.

2.3 Singular Value Decomposition

Singular Value Decomposition (SVD) is a very powerful technique which can deal with a set of linear equations or matrices that are either singular or numerically very close to singular [5]. By implementation of SVD, $\mathbf{E} \in \Re^{n\times n}$ can be written as the product of a column-orthogonal matrix $\mathbf{U} \in \Re^{n\times n}$, a diagonal matrix $\mathbf{\Lambda} \in \Re^{n\times n}$ with nonnegative elements, and the transpose of a column-orthogonal matrix $\mathbf{V} \in \Re^{n\times n}$:

$$\mathbf{E} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^\top \tag{11}$$

where

$$\begin{aligned} \mathbf{U}^\top\mathbf{U} &= \mathbf{I} \\ \mathbf{V}^\top\mathbf{V} &= \mathbf{I} \end{aligned} \tag{12}$$

and $\mathbf{I} \in \Re^{n\times n}$ denotes the identity matrix. The diagonal elements of $\mathbf{\Lambda}$ are called *singular values*. If the rank of $\mathbf{E}$ is r, then it has r positive and $n - r$ additional zero singular values. The positive singular values, $\lambda_1, \lambda_2, \ldots, , \lambda_r$, are numbered in non-increasing order as

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0 \tag{13}$$

Since setting the smallest positive singular value of $\mathbf{E}$ to zero may lead to minimal modification of it, we set the last smallest $h^\star$ singular values of $\mathbf{E}$ to zero to make it have the required rank deficiency:

$$\lambda_{n-h^\star+1} = \lambda_{n-h^\star+2} = \ldots = \lambda_n = 0 \tag{14}$$

Let $\bar{\mathbf{\Lambda}}$ denote the updated $\mathbf{\Lambda}$ so that it has $h^{\star}$ null diagonal terms. The updated version $\bar{\mathbf{E}}$ of the equilibrium matrix that has $h^{\star}$ rank deficiency can then be derived as follows

$$\bar{\mathbf{E}} = \mathbf{U}\bar{\mathbf{\Lambda}}\mathbf{V}^{\top} \tag{15}$$

2.4 Form-finding Process

A new force density vector $\bar{\mathbf{q}}$ can then be achieved by Eq. (10) with the new $\bar{\mathbf{E}}$ that is incorporated to compute $\bar{\mathbf{g}}$. However, since the least square solution is an approximate solution, $\bar{\mathbf{q}}$ may not satisfy the requirement on rank deficiency of the equilibrium matrix exactly in only one step. The following algorithm finds the feasible set of force densities by iteration

Algorithm 1: Feasible force densities

Step 0: Specify an initial force density vector $\mathbf{q}^0$ to obtain $\mathbf{E}^0$ by Eq. (2). Formulate $\bar{\mathbf{B}}$ and $\bar{\mathbf{g}}^0$ with the specified linear constraints. Set $i := 0$.

Step 1: Set the $h^{\star}$ smallest singular values of $\mathbf{E}^i$ to zero to reconstruct $\bar{\mathbf{E}}^i$.

Step 2: Obtain $\bar{\mathbf{g}}^{i+1}$, calculate $\mathbf{q}^{i+1}$ and update $\mathbf{E}^{i+1}$.

Step 3: Check the rank of $\mathbf{E}^{i+1}$, i.e. if it satisfies $n - \mathrm{rank}(\mathbf{E}^{i+1}) = h^{\star}$, then let $\hat{\mathbf{q}} = \mathbf{q}^{i+1}$ and terminate the algorithm; otherwise, set $i \longleftarrow i+1$ and return to Step 1.

This way, we can obtain the force density vector $\hat{\mathbf{q}}$, which ensures the equilibrium matrix $\hat{\mathbf{E}}$ has the required rank deficiency of $h^{\star}$.
Let $\mathbf{H} \in \Re^{3n\times 3n}$ denote the tensor product of the identity matrix $\mathbf{I} \in \Re^{3\times 3}$ and $\hat{\mathbf{E}}$ as

$$\mathbf{H} = \mathbf{I} \otimes \hat{\mathbf{E}} \tag{16}$$

The equilibrium equations in all directions can be combined as follows:

$$\mathbf{H}\mathbf{X} = \mathbf{0} \tag{17}$$

The solution of Eq. (17) can be written as

$$\mathbf{X} = \mathbf{P}\boldsymbol{\beta} \tag{18}$$

where $\mathbf{P} \in \Re^{3n\times 3h^{\star}}$ satisfies $\mathbf{HP} = \mathbf{0}$, and $\boldsymbol{\beta} \in \Re^{3h^{\star}}$ is the coefficient vector. By specifying an independent set of nodal coordinates $\bar{\mathbf{X}}$, configuration of the structure can be uniquely determined by

$$\mathbf{X} = \mathbf{P}\bar{\mathbf{P}}^{-1}\bar{\mathbf{X}} \tag{19}$$

An algorithm that enables designers to specify the set of independent nodal coordinates consecutively can be found in [6].
The design procedure of the proposed adaptive force density method can be summarized as follows

Algorithm 2: Design Procedure

Step 1: Formulate the linear constraints on force densities, and specify an initial set of force densities.

Step 2: Obtain a feasible set of force densities by implementation of Algorithm 1.

Step 3: Specify an independent set of nodal coordinates to obtain a unique configuration of the structure.

3. NUMERICAL EXAMPLES

The proposed method is implemented by MATLAB [7]. The good convergence of Algorithm 1 and the ability of searching new configurations by changing the values of parameters are illustrated by some numerical examples in the section.

3.1 Two-stage self-stressed structure

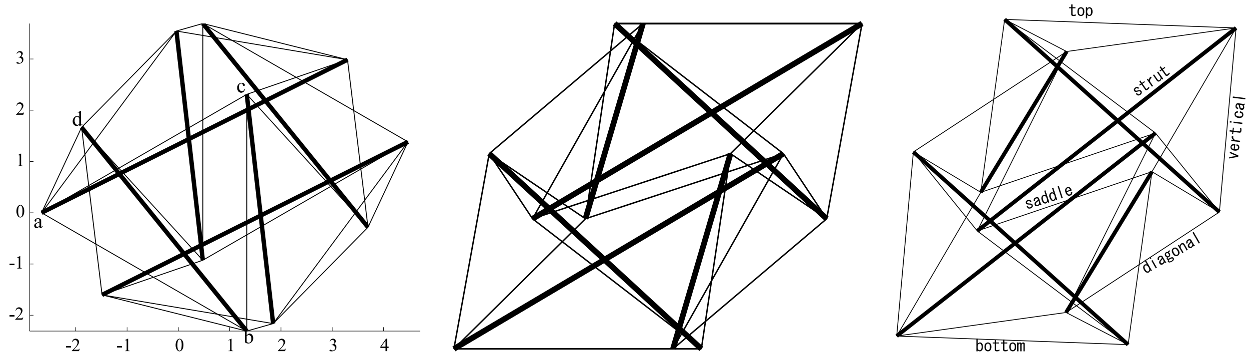
First, we consider a two-stage self-stressed structure, which is composed of (n=)12 nodes and (m =)30 members, six struts and 24 cables. For simplicity, struts of the structure are divided into two groups (1) upper stage and (2) lower stage; and its cables are divided into: (3) top and bottom bases, (4) saddle, (5)

Table 1: Force densities of Example 1.

Group	1	2	3	4	5	6
Initial Value	-1.5	-1.5	1	2	1	1
Finial Value	-1.8376	-1.8376	0.9281	1.9918	1.1737	0.9958

Table 2: Specified nodal coordinates in Example 1.

Node	a	b	c	d
x-direction	-2.6667	1.3333	1.3334	-1.8867
y-direction	0	-2.3094	2.3094	1.6666
z-direction	0	0	0	3.3333



(a) top view (b) side view (c) perspective view

Fig. 2: Example 1: a two-stage self-stressed structure (type 1).

vertical, and (6) diagonal as described in [8]. A two-stage and its groups are shown in Fig. 2.(c), where thin and thick lines denote cables and struts, respectively.

Example 1: If the initial set of force densities is specified as listed in Table 1, we obtain the feasible set of force densities as listed in the last row of Table 1. In this case, all force densities of the same group have the same values. Specifying the coordinates of three independent nodes a, b, c defined in Fig. 2(a) to locate the bottom base on the $xy-$plane, and node d in the lower stage as listed in Table 2, we achieve the configuration of the structure as shown in Fig. 2. It can be observed in this example that the members of each group are rotationally symmetric, although it may not be always the case.

In Example 1, Algorithm 1 implements 158 iterations to obtain the required rank deficiency of the equilibrium matrix. The relative error of force density vector at each iteration, defined as the Euclidean norm $||\mathbf{q}^i - \hat{\mathbf{q}}||$ of the difference of $\mathbf{q}^i$ at ith iteration to the final one $\hat{\mathbf{q}}$ is plotted in Fig. 3. A very good convergence of the proposed algorithm can be seen from the Fig. 3 that the relative error comes very close to zero by only 20 iterations.

It is clear from the design procedure of Example 1 that we can obtain new configurations by modifying the values of initial force densities in the first step and/or the independent nodal coordinates in the last step.

Example 2: If the initial value of force density of the struts in the upper stage is modified from -1.5 in Example 1 to -0.7 as Example 2, and use the same values for other parameters, including force densities and nodal coordinates, it may be interesting to see that the struts in the upper stage becomes shorter than the struts in the lower stage.

Example 3: If we change only the initial force density for the group of saddle cables from 2 in Example 1 to 0.5 as Example 3, and do not modify the nodal coordinates, the configuration is found as shown in Fig. 5. It should be noticed that there is no pair of parallel struts in this case.

Example 4: Besides changing the initial force densities at the first step of the form-finding process, we can modify the nodal coordinates at the last step as well to search for new configurations. If we modify the y-coordinate of node a from -1.8867 in Example 1 to -3.8867 as Example 4, and remain the other specified values untouched as Example 3. The obtained structure as shown Fig. 6 becomes fatter comparing to the structure in Example 1.

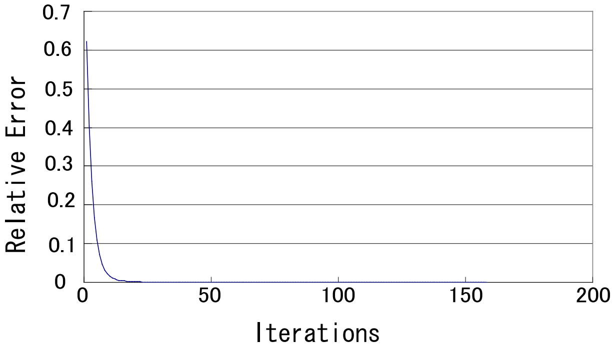
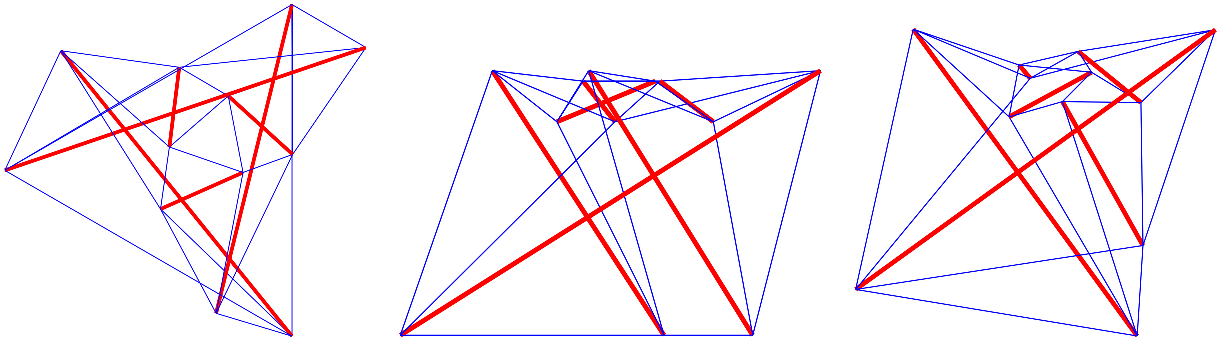
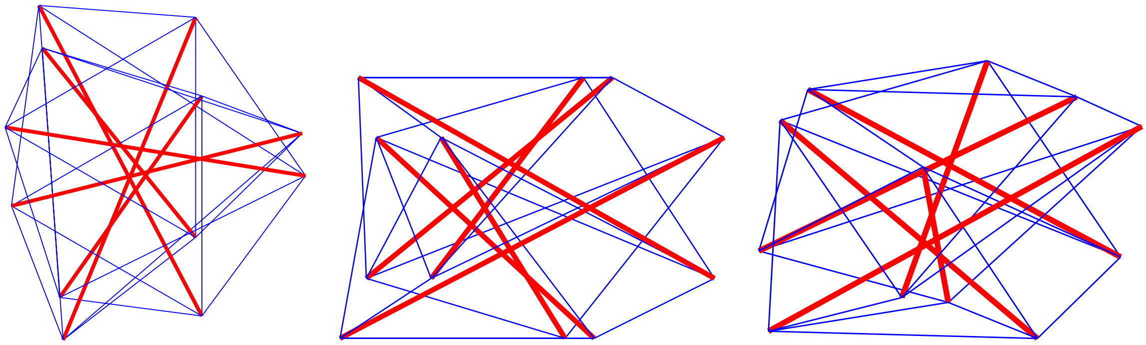


Fig. 3: Convergence of Algorithm 1 for feasible force densities.



(a) top view (b) side view (c) perspective view

Fig. 4: Example 2: a two-stage self-stressed structure (type 2).



(a) top view (b) side view (c) perspective view

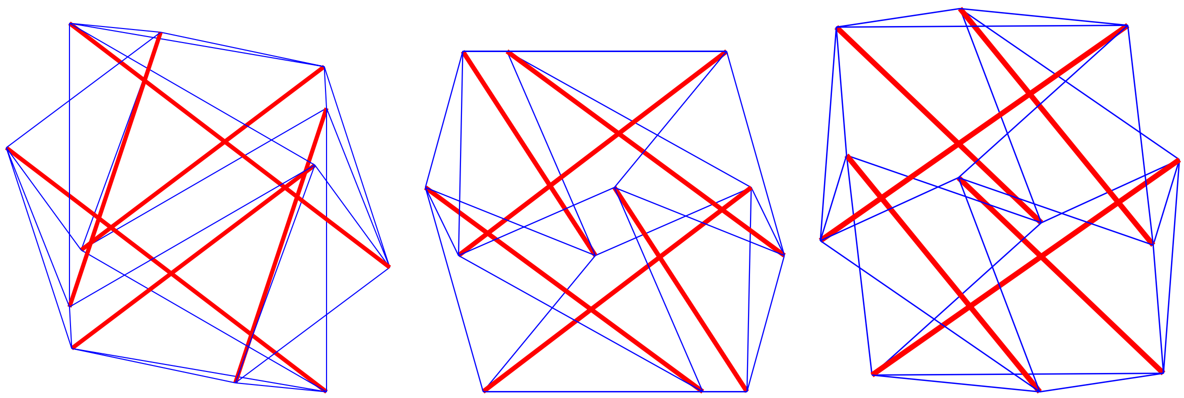
Fig. 5: Example 3: a two-stage tensegrity structure (type 3).

3.2 Three-stage self-stressed structure

Example 5: A three-stage self-stressed structure consists of (m =)48 members, including 9 struts and 39 cables, and (n =)18 nodes. The struts of a three-stage self-stressed structure are classified into two groups: (1) the six struts in the upper and lower stages, and (2) the three struts in the center stage. The cables are classified into the same groups as a two-stage structure.

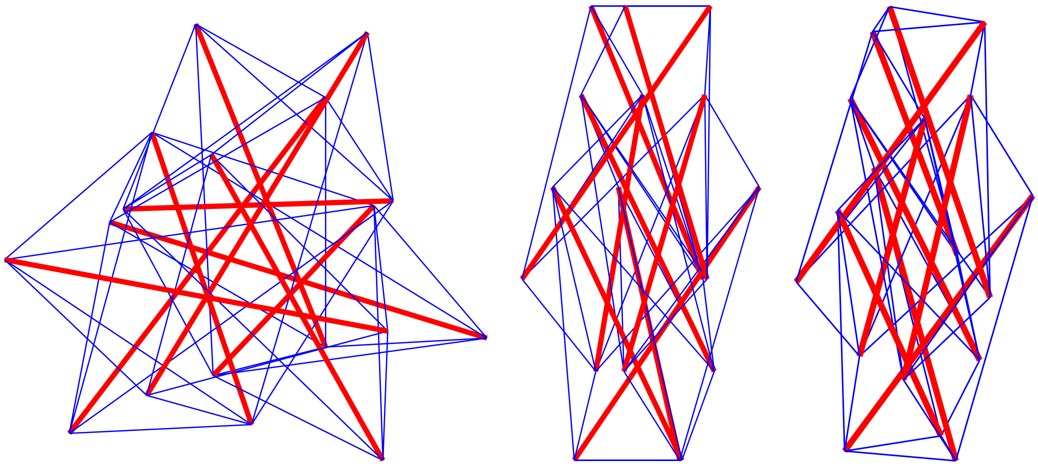
Set the initial set of force densities as {-0.8, -1, 0.8, 0.3, 0.7, 0.5}, and specify the nodal coordinates of the bottom base and a lower node in the central stage as $\{(-2.6667, 0, 0), (1.3333, -2.3094, 0), (1.3334, 2.3094, 0)\}$ and $(-1.8867, 1.6666, 3.3333)$, respectively. We can obtain the configuration of a three-stage self-stressed structure as shown in Fig. 7.

Stability of the obtained structures in the above numerical examples have been confirmed by checking the signs of the eigenvalues of their tangent stiffness matrices where the rigid-body motions have been constrained; i.e. if all the eigenvalues are positive the structure is stable, otherwise, the structure is unstable.



(a) top view (b) side view (c) perspective view

Fig. 6: Example 4: a two-stage self-stressed structure (type 4).



(a) top view (b) side view (c) perspective view

Fig. 7: Example 5: three-stage self-stressed structure.

4. DISCUSSIONS AND CONCLUSIONS

A numerical method for the form-finding problem of self-stressed structures is extended from the basic idea and formulations of force density method initial developed for cable nets. Based on singular value decomposition of the equilibrium matrix, an algorithm is presented for finding a set of force densities, which satisfies the requirement on rank deficiency of the equilibrium matrix, by setting a specific number of smallest singular values to zero. After specifying an independent set of nodal coordinates, we can obtain a unique geometrical configuration of the structure.

The good convergence of the proposed method is illustrated by a numerical example. To find a unique configuration, only topology, initial force densities given at the first step and an independent set of nodal coordinates specified at the last step are required. The proposed approach has also a very strong ability of systematically searching new geometrical configurations by changing the initial force densities and nodal coordinates.

However, it may not have exact and direct control over the geometrical and mechanical properties of the structure, since the parameters in the family of force density method are neither forces nor lengths but the force-to-length ratios. As mentioned in [3], less symmetric or even asymmetric configurations may be found for a given symmetric force densities, because the member lengths cannot be described explicitly and linearly in the formulation.

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