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Reprinted from

INTERNATIONAL JOURNAL OF
SPACE STRUCTURES
Volume 21 · Number 4 · 2006

MULTI-SCIENCE PUBLISHING CO. LTD.
5 Wates Way, Brentwood, Essex CM15 9TB, United Kingdom

Form-finding of Tensegrity Structures Subjected to Geometrical Constraints

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ABSTRACT: Geometrical constraints are formulated as linear equations with respect to force densities and nodal coordinates. These constraints are incorporated into the adaptive force density method proposed by the authors for finding the non-degenerate and self-equilibrated configurations of tensegrity structures. The form-finding process is divided into two design stages: (1) finding the feasible force densities that satisfy the non-degeneracy condition, and (2) uniquely determining the configuration (nodal location) that satisfies the self-equilibrium conditions. The proposed method is demonstrated to be powerful for systematically searching new configurations of tensegrity structures subjected to given geometrical constraints.

Key words: Tensegrity Structure; Form-finding; Geometrical Constraint; Force Density Method.

1. INTRODUCTION

This paper presents an efficient numerical method for finding the self-equilibrated configurations of tensegrity structures. The process of finding such configurations satisfying self-equilibrium equations is called *form-finding*. We will show in this paper that the final configurations determined by the proposed method precisely have the geometrical properties specified by the users.

The word *tensegrity* introduced by Fuller [1] is a contraction of ‘tension’ and ‘integrity’. It indicates the stabilizing effect by the introduction of member forces (tension and compression) to the structure. In the original and strict definition of tensegrity structures, every strut carrying compression does not make mechanical contact with any other struts. For example, Fig.1 shows the simplest tensegrity structure in three-dimensional space. It consists of six nodes and twelve members; the members in thick lines are struts that

transmit only compressive forces, and the members in thin lines are cables that can transmit only tensile forces. Since no strut contacts with other struts, every node of the structure is physically connected with only one strut.

Tensegrity structures belong to the Type III pin-jointed structures according to the following classification for general pin-jointed structures based on their stability properties [2]:

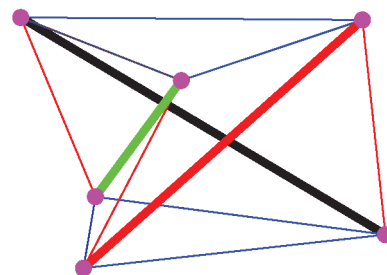


Fig. 1. The simplest three-dimensional tensegrity structure.

- Type I: truss, which contains no self-stress.
- Type II: tensile structure, all members of which are in tension.
- Type III: tensegrity structure, which consists of both tensile members (cables) and compressive members (struts).

The force density method [3] has great advantages in transforming the non-linear self-equilibrium equations into a set of linear equations by introducing the concept of force density, which is defined as force-to-length ratio. It can be easily applied to the form-finding problems of tensile structures because the force density matrix is positive definite in these cases.

However, difficulties arise in applying the force density method to the form-finding problem of tensegrity structures: the associated force density matrix should have necessary rank deficiency for a non-degenerate structure, because there exist compressive members and this kind of structures are free-standing. For this purpose of ensuring the force density matrix to have the necessary rank deficiency in the form-finding process, an analytical method [4] and a numerical method [5] were presented.

When the structure is relatively simple, the analytical method can find the relationship among the feasible force densities so as to provide a deeper insight into the mechanical properties of the structure. However, when the structure is relatively complex with many nodes and members, the analytical method might not be efficient enough and the relationship among a large number of unknown force densities may turn out to be less meaningful. For this case, the numerical method, called *adaptive force density method*, is considered to be a more powerful tool for systematically searching for new configurations.

However, both of the above-mentioned methods cannot have control over the geometrical properties of a tensegrity structure with precision, e.g. asymmetric configuration may be obtained although it is expected to be symmetric. This is because the geometrical constraints have not yet been completely included in the form-finding process.

In this paper, the adaptive force density method is extended and divided into two subsequent design stages to directly control the self-equilibrated configurations of tensegrity structures. The first design stage is to find the feasible force densities satisfying the non-degeneracy condition, and the second stage is to determine the self-equilibrated configuration satisfying the equilibrium equations.

Geometrical constraints are formulated in linear forms with respect to the force densities and nodal coordinates, and respectively incorporated into these two design stages. As examples to demonstrate how the proposed form-finding process works, rotational symmetry about z-axis and constraint on elevation (z-coordinates) are considered.

The paper following this introduction is organized as follows: Section 2 briefly gives the self-equilibrium equations and the non-degeneracy condition of a tensegrity structure. Section 3 presents the formulations of the geometrical constraints on the structure, including rotational symmetry and constraints on elevation. The geometrical constraints are incorporated with the adaptive force density method in Section 4. Section 5 gives a detailed description on the geometrical properties of tensegrity towers, which are used as examples to illustrate the validity and capability of the proposed method in Section 6. Section 7 discusses the advantages and disadvantages of the proposed method and concludes the study.

2. SELF-EQUILIBRIUM EQUATIONS AND NON-DEGENERACY CONDITION

Let m and n denote the numbers of members and nodes, respectively. Topology of a tensegrity structure can be defined as a directed graph [6], and described by the connectivity (incidence) matrix $\mathbf{C} \in \mathfrak{R}^{m \times n}$: if member k is connected to nodes i and j ($i < j$), then the i th and j th elements of the k th row of \mathbf{C} are equal to $+1$ and -1 , respectively, while other elements in the row are zero. Note that \mathbf{C} is a constant matrix when the topology is given.

Let \mathbf{x} , \mathbf{y} and \mathbf{z} ($\in \mathfrak{R}^n$) denote the nodal coordinate vectors in x -, y - and z - directions, respectively. Denote the force density q_k of member k as the force s_k to length l_k ratio; i.e., $q_k = s_k/l_k$. Since a strut and a cable can transmit only compressive and tensile forces, respectively, we have $q_k < 0$ for a strut and $q_k > 0$ for a cable. The force density vector consisting of q_k ($k = 1, \dots, m$) is denoted by $\mathbf{q} \in \mathfrak{R}^m$.

When no external load is applied, the structure is in a state of self-equilibrium, and the self-equilibrium equations with respect to the nodal coordinate vectors \mathbf{x} , \mathbf{y} and \mathbf{z} in each direction can be respectively written as [5]

$$\mathbf{E}\mathbf{x} = \mathbf{0} \quad (1.1)$$

$$\mathbf{E}\mathbf{y} = \mathbf{0} \quad (1.2)$$

$$\mathbf{E}\mathbf{z} = \mathbf{0} \quad (1.3)$$

where the *force density matrix* $\mathbf{E} \in \mathfrak{R}^{n \times n}$ can be formulated by using the connectivity matrix \mathbf{C} and the diagonal version of the force density vector \mathbf{q} as

$$\mathbf{E} = \mathbf{C}^T \text{diag}(\mathbf{q})\mathbf{C} \tag{2}$$

\mathbf{E} is non-linear with respect to the nodal coordinates because member lengths are involved. Hence, the self-equilibrium equations (1.1)-(1.3) are also non-linear with respect to the nodal coordinates. In the case where the force densities are known or given, \mathbf{E} becomes a constant matrix, so that the originally non-linear equations are transformed into a set of linear equations with respect to the nodal coordinate vectors \mathbf{x} , \mathbf{y} and \mathbf{z} , respectively. This is the basic idea of the force density method.

From the definition in Eq. (2), the force density matrix \mathbf{E} is always square and symmetric, and furthermore, it has rank deficiency of at least one, because the sum of the components in each row or column is 0. Let r denote the rank deficiency of \mathbf{E} as

$$r = n - \text{rank}(\mathbf{E}) \tag{3}$$

The solution space of nodal coordinates in each direction can then be described by a linear combination of r independent vectors lying in the null-space of \mathbf{E} . As discussed in [5] based on dimensions of the solution space, the following condition should be satisfied for a non-degenerate structure in d -dimensional ($d=2$ or 3) space

$$r \geq r^* = d + 1 \tag{4}$$

where r^* denotes the necessary rank deficiency of \mathbf{E} .

When Eq. (4) is not satisfied, the structure in d -dimensional space will degenerate into a space with lower dimensions, such as a plane ($r = 3$), a line ($r = 2$) or even a point ($r = 1$). Hence, the problem of finding the configuration of a non-degenerate tensegrity structure in three-dimensional space turns out to that of finding the feasible force densities that satisfy Eq. (4) with $d = 3$.

It should be noted that this is only the necessary but not sufficient condition for a non-degenerate tensegrity structure; i.e., the structure satisfying Eq. (4) may still be degenerate if its nodal coordinates in each direction are linearly dependent [7].

3. GEOMETRICAL CONSTRAINTS

In this section, the geometrical constraints, such as rotational symmetry and constraint on elevation (z -coordinates), are formulated as linear equations with respect to the force densities and nodal coordinates, respectively; they will be respectively incorporated into the two separate design stages of the proposed method. Besides the constraints presented in this section, any other geometrical constraints can also be incorporated in the proposed method, as long as they can be formulated as linear equations with respect to the force densities and nodal coordinates.

3.1 Rotational Symmetry and Force Densities

We use the term – n^b -fold rotational symmetry – to indicate the fact that a member among a set with n^b members can be moved to any other in the set by rotation about z -axis through an angle $2i\pi/n^b$, where i is an appropriate integer value. These n^b members are said to be in the same orbit; they have the same lengths and forces, and accordingly, the same force densities.

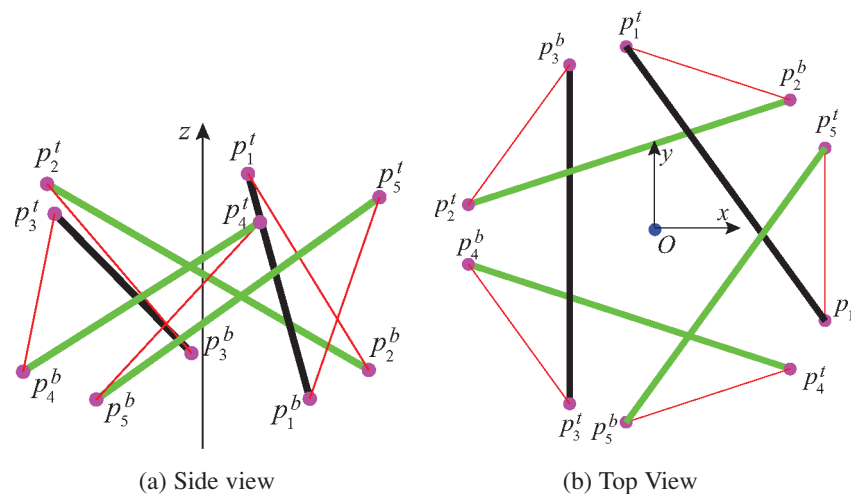


Fig. 2. An example of rotational symmetry about z -axis of five struts.

For example, the five struts shown in thick lines in Fig.2 belong to the same orbit. Any one of the struts can be moved to another by rotation about the z -axis through a proper angle $2i\pi/5$ ($i \in \{1, 2, 3, 4\}$). The cables in thin lines belong to a different orbit with the same 5-fold rotational symmetry.

Since the members in the same orbit have the same force densities, we have the following relation

$$\mathbf{F}\mathbf{q} = \mathbf{0} \quad (5)$$

where there are only two non-zero elements $+1$ and -1 in each row of \mathbf{F} in the proper positions. For example, if members i and j ($i < j$) are in the same orbit, there is one row k of \mathbf{F} , of which the i th and j th elements are $+1$ and -1 , respectively, while the remaining elements in the row are 0:

$$\mathbf{F}_{(k,p)} = \begin{cases} 1 & \text{for } p = i \\ -1 & \text{for } p = j \\ 0 & \text{for other cases} \end{cases} \quad (6)$$

The linear constraint (5) on force densities will be incorporated into the first design stage for finding the feasible force densities.

3.2 Rotational Symmetry and Nodal Coordinates

Since every node of a tensegrity structure is connected to only one strut in the strict definition of tensegrity structures, it is sufficient to consider only the struts in describing the symmetry properties of the structure. Moreover, since the struts in different orbits are geometrically independent in view of symmetry, the symmetry properties can be formulated only for the struts in one orbit, and then be simply extended to the whole structure.

Consider a symmetric tensegrity structure with n^l orbits of struts and n^b struts in each orbit. Denote the higher and lower nodes of the strut i by p_i^t and p_i^b , respectively. The x - and y -coordinates of the nodes p_i^t and p_i^b are denoted by the vectors $\mathbf{x}_i^t = (x_i^t, y_i^t)^\top$ and $\mathbf{x}_i^b = (x_i^b, y_i^b)^\top$, respectively.

The directed member vector $\mathbf{d}_i \in \mathfrak{R}^2$ ($i = 1, \dots, n^b$) of the struts on xy -plane as

$$\mathbf{d}_i = \mathbf{x}_i^b - \mathbf{x}_i^t \quad (7)$$

which corresponds to the edges of a directed graph [6]. For example, Fig.3 shows the directed graph of the five struts with 5-fold rotational symmetry in Fig. 2.

\mathbf{d}_i in the k th orbit of struts of the structure are combined to $\mathbf{d}^k \in \mathfrak{R}^{2n^b}$ as

$$\mathbf{d}^k = (\mathbf{d}_1^\top, \dots, \mathbf{d}_{n^b}^\top)^\top \quad (8)$$

\mathbf{x}_i^b and \mathbf{x}_i^t in the orbit are also combined to $\mathbf{x}^k \in \mathfrak{R}^{4n^b}$ as

$$\mathbf{x}^k = ((\mathbf{x}_1^b)^\top, \dots, (\mathbf{x}_{n^b}^b)^\top, (\mathbf{x}_1^t)^\top, \dots, (\mathbf{x}_{n^b}^t)^\top)^\top \quad (9)$$

The relationship between \mathbf{d}^k and \mathbf{x}^k in orbit k can be written as follows by a matrix $\mathbf{T}^k \in \mathfrak{R}^{2n^b \times 4n^b}$:

$$\mathbf{d}^k = \mathbf{T}^k \mathbf{x}^k \quad (10)$$

where \mathbf{T}^k is constructed by the $2n^b$ -by- $2n^b$ identity matrix \mathbf{I}^{2n^b} as

$$\mathbf{T}^k = (\mathbf{I}^{2n^b}, -\mathbf{I}^{2n^b}) \quad (11)$$

The transformation matrix \mathbf{R}_i for the positive rotation about z -axis through the angle $\theta_i = 2(i - 1)\pi/n^b$ is given as

$$\mathbf{R}_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \quad (12)$$

If \mathbf{d}_1 coincides with \mathbf{d}_i by the a positive rotation about z -axis through θ_i , the following relation holds:

$$\mathbf{d}_i = \mathbf{R}_i \mathbf{d}_1 \quad (13)$$

Eq. (13) can be rewritten as follows by using the 2-by-2 identity matrix \mathbf{I}^2 :

$$(\mathbf{R}_i, \dots, -\mathbf{I}^2, \dots) \mathbf{d}^k = \mathbf{0} \quad (14)$$

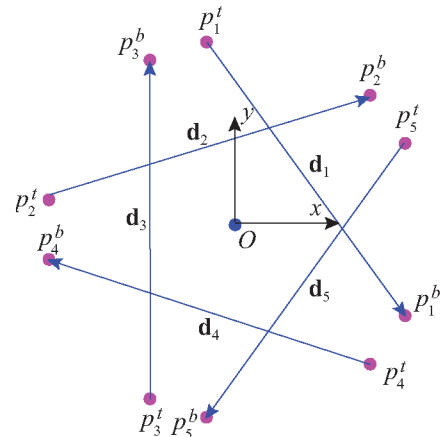


Fig. 3. Directed graph of the five struts in Fig.2.

Combining all the relations of \mathbf{d}_i ($i \neq 1$) and \mathbf{d}_1 similar to Eq. (14) by using the matrix $\mathbf{S}^k \in \mathfrak{R}^{2(n^b-1) \times 2n^b}$, we obtain

$$\mathbf{S}^k \mathbf{d}^k = \mathbf{0} \quad (15)$$

By substituting Eq. (10) into Eq. (15) and letting $\bar{\mathbf{S}} = \mathbf{S}^k \mathbf{T}^k \in \mathfrak{R}^{2(n^b-1) \times 4n^b}$, the rotational symmetry of the struts in orbit k can be expressed in a linear form with respect to their nodal coordinates in xy -plane as

$$\bar{\mathbf{S}} \mathbf{x}^k = \mathbf{0} \quad (16)$$

Because symmetry of the struts in different orbits are independent, rotational symmetry of the whole structure in terms of the generalized coordinate vector $\mathbf{X} = ((\mathbf{x}^1)^\top, \dots, (\mathbf{x}^{n^l})^\top)^\top \in \mathfrak{R}^{2n}$ in xy -plane can then be written as

$$\mathbf{S} \mathbf{X} = \mathbf{0} \quad (17)$$

where $\mathbf{S} \in \mathfrak{R}^{2n^l(n^b-1) \times 4n^l n^b}$ is the tensor product of the n^l -by- n^l identity matrix \mathbf{I}^{n^l} and the matrix $\bar{\mathbf{S}}$; i.e., $\mathbf{S} = \mathbf{I}^{n^l} \otimes \bar{\mathbf{S}}$.

This way, rotational symmetry of the whole structure is formulated as a linear equation with respect to the generalized nodal coordinate vector \mathbf{X} in xy -plane, which will be incorporated in the second design stage of the proposed method to determine the final configuration.

3.3 Elevation Constraints

Suppose that the elevation (z -coordinates of the structure) of the structure is assigned by the designer. Hence, z -coordinates \mathbf{z} of all the nodes are determined a priori, and we have to assure that the final configuration of the structure has precisely these given z -coordinates.

Since the relation

$$\text{diag}(\mathbf{q}) \mathbf{C} \mathbf{z} = \text{diag}(\mathbf{C} \mathbf{z}) \mathbf{q} \quad (18)$$

always holds, the self-equilibrium equation (1.3) in z -direction can be rewritten with respect to \mathbf{q} as

$$\mathbf{C}^\top \text{diag}(\mathbf{C} \mathbf{z}) \mathbf{q} = \mathbf{0} \quad (19)$$

By letting $\mathbf{H} = \mathbf{C}^\top \text{diag}(\mathbf{C} \mathbf{z})$, the geometrical constraints on the elevation of a tensegrity structure with respect to \mathbf{q} can then be written as

$$\mathbf{H} \mathbf{q} = \mathbf{0} \quad (20)$$

Note that \mathbf{H} is a constant matrix, because \mathbf{C} is given and \mathbf{z} is assigned. Eq. (20) will be incorporated in the first design stage in the next section for finding the feasible force densities.

4. FORM-FINDING PROCESS

In the form-finding process for tensegrity structures in this study, force densities and nodal coordinates are unknown parameters that need to be determined. The proposed method is divided into two subsequent design stages: (1) finding the feasible force densities that satisfy the non-degeneracy condition, and (2) determining the configuration. The geometrical constraints in linear forms with respect to force densities and nodal coordinates formulated in the previous section will be respectively incorporated into these two design stages.

4.1 Feasible Force Densities (First Design Stage)

In the first design stage of the form-finding process, an iterative algorithm is presented to find the feasible force densities satisfying the non-degeneracy condition (4) and the geometrical constraints (21) and (22).

4.1.1 Constraints on Force Densities

The force density matrix \mathbf{E} is defined by the given connectivity \mathbf{C} and the unknown force densities \mathbf{q} . If we assemble the columns of \mathbf{E} as a vector $\mathbf{g} \in \mathfrak{R}^{n^2}$, then \mathbf{q} and \mathbf{g} can be written in a linear form as follows through a constant matrix $\mathbf{B} \in \mathfrak{R}^{n^2 \times m}$ [5]

$$\mathbf{B} \mathbf{q} = \mathbf{g} \quad (21)$$

In each row of \mathbf{B} , there are only two nonzero elements +1 and -1.

So far, we have three linear equations (5), (20) and (21) with respect to \mathbf{q} , which are related to rotational symmetry, constraint on elevation and the force density matrix, respectively. The geometrical constraints (5) and (20) are combined as

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{H} \end{pmatrix} \mathbf{q} = \mathbf{A} \mathbf{q} = \mathbf{0} \quad (22)$$

If the rank of the matrix \mathbf{A} in Eq. (22) is equal to the number of members m , the geometrical constraints should be relaxed because there exists no non-trivial solution satisfying these constraints. If its rank is less than m , then the solution of \mathbf{q} can be written as

$$\mathbf{q} = \Psi \alpha \quad (23)$$

where α is a coefficient vector, and the columns of the matrix Ψ span the solution space of Eq. (22). Note that Ψ is also a constant matrix when the constraints are given. Since the force densities define \mathbf{E} , which has to satisfy the non-degeneracy condition (4), the coefficient vector α cannot be selected arbitrarily. From Eqs. (21) and (23) we have

$$\mathbf{g} = \mathbf{B}\Psi\alpha \quad (24)$$

Since $\mathbf{B}\Psi$ in Eq. (24) is usually full-rank and not square, α can be computed as follows by using the least square method

$$\alpha = (\mathbf{B}\Psi)^{-}\mathbf{g} \quad (25)$$

where $()^{-}$ denotes the generalized inverse matrix.

Suppose that we have a force density matrix \mathbf{E}^i with the necessary rank deficiency $(r^i =)r^*$ at the i th step of the iterative algorithm. \mathbf{E}^i is calculated by Eq. (2) with its corresponding force density vector \mathbf{q}^i . The elements of \mathbf{E}^i are assembled to construct the vector \mathbf{g}^i . From Eqs. (23) and (25), we have a new version of the force density vector \mathbf{q}^{i+1} as follows

$$\mathbf{q}^{i+1} = \Psi(\mathbf{B}\Psi)^{-}\mathbf{g}^i \quad (26)$$

Note that \mathbf{q}^{i+1} may not be equal to \mathbf{q}^i , so \mathbf{E}^{i+1} corresponding to \mathbf{q}^{i+1} may not have the necessary rank deficiency and has to be recomputed based on the eigenvalue analysis and spectral decomposition introduced below.

4.1.2 Algorithm for Feasible Force Densities

The force density matrix \mathbf{E}^i is symmetric and can be decomposed as [8]

$$\mathbf{E}^i = \Phi^i\Lambda^i(\Phi^i)^T \quad (27)$$

where columns of $\Phi^i \in \mathfrak{R}^{n \times n}$ are the eigenvectors of \mathbf{E}^i , and the diagonal elements $\{\lambda_1^i, \lambda_2^i, \dots, \lambda_n^i\}$ of the diagonal matrix $\Lambda^i \in \mathfrak{R}^{n \times n}$ are the eigenvalues.

Because the rank deficiency r is defined by the number of zero eigenvalues of \mathbf{E}^i , we can make it have the necessary rank deficiency r^* by artificially assigning r^* zero to the elements in the leading diagonal of Λ^i . The best way to do this is to assign zero to the elements (eigenvalues) with the smallest values. This way, the diagonal matrix Λ^i becomes $\bar{\Lambda}^i$, and the force density matrix is updated as $\bar{\mathbf{E}}^i$:

$$\bar{\mathbf{E}}^i = \Phi^i\bar{\Lambda}^i(\Phi^i)^T \quad (28)$$

It is clear that $\bar{\mathbf{E}}^i$ has the necessary rank deficiency r^* .

Eqs. (26) and (28) are iteratively computed until \mathbf{E}^{i+1} corresponding to \mathbf{q}^{i+1} calculated by Eq. (26) satisfies the necessary non-degeneracy condition. The algorithm of finding these feasible force densities \mathbf{q}^{i+1} is summarized as follows:

Algorithm 1: Feasible force densities

Step 0: Give an initial \mathbf{q}^0 to obtain \mathbf{E}^0 by Eq. (2). Set $i := 0$.

Step 1: Assign zero to the r^* smallest (absolute) eigenvalues of \mathbf{E}^i and reconstruct $\bar{\mathbf{E}}^i$ by Eq. (28).

Step 2: Obtain \mathbf{g}^{i+1} , calculate \mathbf{q}^i from Eq. (26) and update \mathbf{E}^{i+1} by Eq. (2).

Step 3: If Eq. (4) holds, then let $\hat{\mathbf{q}} = \mathbf{q}^{i+1}$, compute $\hat{\mathbf{E}}$ and terminate the algorithm; otherwise, set $i \leftarrow i + 1$ and return to Step 1.

4.2 Configuration (Second Design Stage)

If elevation of the structure is assigned as a geometrical constraint, the z -coordinates of the structure are known a priori. The only unknown parameters are the nodal coordinates in xy -plane, which should satisfy the constraint (17) on rotational symmetry and the self-equilibrium equations (1.1) and (1.2). These constraints are combined to the following linear equation with respect to the nodal coordinates \mathbf{X} in xy -plane

$$\begin{pmatrix} \mathbf{E} \otimes \mathbf{I}^2 \\ \mathbf{S} \end{pmatrix} \mathbf{X} = 0 \quad (29)$$

Eq. (29) should have non-trivial solutions for \mathbf{X} written as follows, otherwise the constraints need to be relaxed

$$\mathbf{X} = \Omega\beta \quad (30)$$

where β is a coefficient vector. Dimension r^c of the vector β indicates that we can arbitrarily specify r^c independent xy -coordinates to determine all the other coordinates of the structure. The specified coordinates are denoted by $\mathbf{X}^c \in \mathfrak{R}^{r^c}$, and the rows of Ω corresponding to \mathbf{X}^c are assembled as $\Omega^c \in \mathfrak{R}^{r^c \times r^c}$. Hence, \mathbf{X} can be computed as

$$\mathbf{X} = \Omega(\Omega^c)^{-1}\mathbf{X}^c \quad (31)$$

This way, configuration of the structure in terms of nodal coordinates can be uniquely determined together with the z -coordinates assigned by the designers.

There can be many ways to specify the independent set of nodal coordinates \mathbf{X}^c . A method for identifying the dependent nodal coordinates based on the Reduced Row-Echelon Form (RREF) of Ω^\top was presented in [9], where the designers specify the independent nodal coordinates one by one. The independent nodal coordinates can also be determined in only one step based on the RREF of Ω^\top , for which a simple example can be found in [5].

4.3 Algorithm Summary

The process of finding the self-equilibrated configuration of a tensegrity structure subjected to geometrical constraints, which can be written in linear forms with respect to force densities and nodal coordinates, is divided into the following two design stages:

Algorithm 2 – Form-finding Process

First Stage: Feasible Force Densities

- (1) Specify the topology.
- (2) Formulate the geometrical constraints in linear forms with respect to force densities.
- (3) Assign an initial set of force densities.
- (4) Find the feasible force densities by Algorithm 1.

Second Stage: Configuration

- (5) Formulate the geometrical constraints in linear forms with respect to nodal coordinates.
- (6) Incorporate the geometrical constraints with respect to nodal coordinates into the self-equilibrium equations.
- (7) Specify an independent set of nodal coordinates

to uniquely determine the self-equilibrated configuration.

In the proposed form-finding process for tensegrity structures, the unknown parameters are force densities and nodal coordinates. As will be demonstrated in the numerical examples, we can search for new configurations of a structure with given topology by changing the values of these parameters in Steps 2, 3, 5 and 7.

We use the value ϵ in the following equation to validate the accuracy of the numerical computations

$$\epsilon = \sqrt{(\mathbf{E}\mathbf{x})^\top(\mathbf{E}\mathbf{x}) + (\mathbf{E}\mathbf{y})^\top(\mathbf{E}\mathbf{y}) + (\mathbf{E}\mathbf{z})^\top(\mathbf{E}\mathbf{z})} \quad (32)$$

When ϵ is small enough, it is considered that the final self-equilibrated configuration determined by the proposed numerical method satisfies the self-equilibrium equations with high accuracy.

5. TENSEGRITY TOWER

Tensegrity tower, e.g., an example structure shown in Fig.4, is a special kind of tensegrity structures. It has one or more layers and at least three struts in each layer. The needle tower invented and built by Kenneth Snelson in Bryant Park in New York may be one of the best-known tensegrity towers. Tensegrity towers will be used as numerical examples in the next section to demonstrate the capability of the proposed method in finding the desired configurations satisfying the geometrical constraints. In this section, a detailed description of the geometrical properties and topology of them is given.

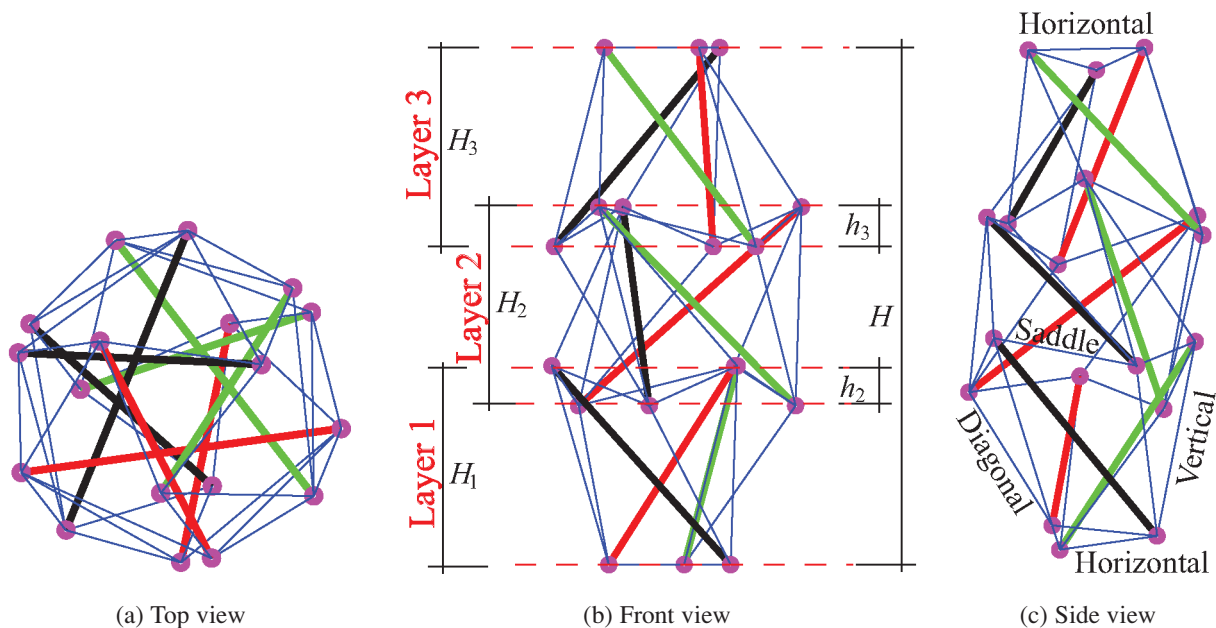


Fig. 4. A symmetric three-layer tensegrity tower with three struts in each layer.

5.1 Configuration

Suppose that a tensegrity tower with n^b -fold rotational symmetry has n^l layers (orbits of struts) and n^b struts in each layer. The struts in each layer belong to the same orbit. The nodes that have the same z -coordinate are said to be in the same plane. Thus, each layer has two different planes – the bottom and the top planes.

Since no strut physically contacts any other strut, the number n of nodes of a tensegrity tower is

$$n = 2n^l n^b \tag{33}$$

The cables of a tensegrity tower are classified into the following four types as shown in Fig.4.(c), based on the connectivity between members and nodes [10]:

- *Horizontal cables* that connect the nodes in the same plane. They can only exist in the bottom plane of the lowest layer and the top plane of the highest layer.
- *Vertical cables* that are connected by the nodes in the top and bottom planes of the same layer.
- *Saddle cables* that connect the nodes in different planes of the adjacent layers, e.g., the top plane of layer k and the bottom plane of layer $k + 1$.
- *Diagonal cables* that connect the nodes in the same top (or bottom) planes of the adjacent layers, e.g., the top (or bottom) plane of layer k and the top (or bottom) plane of layer $k + 1$.

5.2 Elevation

Denote height of the i th layer by H_i ($i = 1, \dots, n^l$), and the overlap between two adjacent layers i and $i-1$ by h_i ($i = 1, \dots, n^l$) where $h_1 = 0$ always holds. The total height H of the structure can be computed by

$$H = \sum_{i=1}^{n^l} (H_i - h_i) \tag{34}$$

The z -coordinates z_k^t of the nodes in the top plane of layer k ($k = 1, \dots, n^l$) can be determined as

$$z_k^t = \sum_{i=1}^k (H_i - h_i) \tag{35}$$

And the coordinates z_k^b of the nodes in the bottom plane of layer k can be computed by

$$z_k^b = z_k^t - H_k \tag{36}$$

This way, z -coordinates of a tensegrity tower are determined.

5.3 Topology

In order to formulate the connectivity matrix \mathbf{C} in a simple manner for a tensegrity tower with any number of layers ($n^l \geq 1$) and any number of struts ($n^b \geq 3$) in each layer, topology of a general tensegrity tower is defined in this subsection.

The nodes in the bottom and top planes of layer k are respectively labelled by $p_{k,j}^b$ and $p_{k,j}^t$ as

$$\begin{aligned} p_{k,j}^b &= 2(k-1)n^b + j \\ p_{k,j}^t &= (2k-1)n^b + j \end{aligned} \quad (j = 1, \dots, n^b) \tag{37}$$

5.3.1 Struts

The j th strut $P_{k,j}^B$ in layer k is connected by nodes $p_{k,j}^b$ and $p_{k,j}^t$ in different planes:

$$P_{k,j}^B = [p_{k,j}^b, p_{k,j}^t] \tag{38}$$

where $[i, j]$ indicates that nodes i and j are connected to construct a member. A simple example with $n^b = 3$ is illustrated in Fig.5, where the vertical and diagonal cables are removed for clarity.

5.3.2 Horizontal and Saddle Cables

For the one-layer tensegrity towers ($n^l = 1$), e.g., the structure as shown in Fig.1, it has been proved that the structure is guaranteed to be stable, irrespective of selection of materials and level of self-stresses, if and only if the horizontal cables are connected by the adjacent nodes [11]. Although the structures can still be possible to be stable in the other cases as discussed in [12], the horizontal and saddle cables are assumed to connect by the adjacent nodes as shown in Fig.5 in this study, in order to avoid the risk of generating an unstable structure. Thus, the connectivity of horizontal and saddle cables are given as follows:

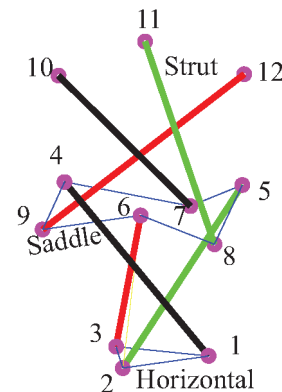


Fig. 5. An example of connectivity of struts, horizontal cables and saddle cables.

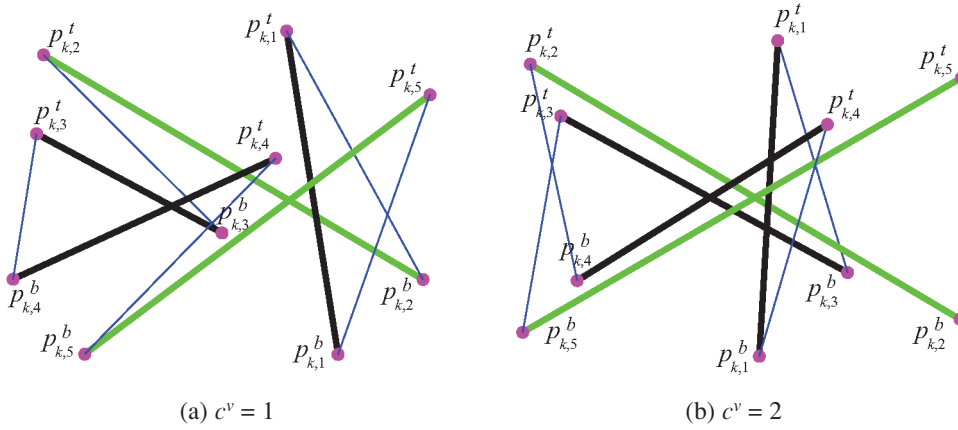


Fig. 6. An example of connectivity of vertical cables in layer k .

- horizontal cables $P_{k,j}^H$ and $P_{n^l,j}^H$ are connected by the adjacent nodes in the bottom and top planes of the lowest and highest layers, respectively, as

$$\begin{aligned} P_{1,j}^H &= [p_{1,j}^b, p_{1,j+1}^b] \\ P_{n^l,j}^H &= [p_{n^l,j}^b, p_{n^l,j+1}^b] \end{aligned} \quad (j = 1, \dots, n^b) \quad (39)$$

- saddle cables are connected by the nodes in the top plane of layer k and the bottom plane of layer $k + 1$ as

$$\begin{aligned} P_{k,2j-1}^S &= [p_{k,j}^t, p_{k+1,j}^b] \\ P_{k,2j}^S &= [p_{k+1,j}^b, p_{k,j+1}^t] \end{aligned} \quad (40)$$

where $j = 1; \dots, n^b$ and $k = 1, \dots, n^l - 1$.

The following relations have been used in Eqs. (39) and (40) for brevity.

$$p_{k,n^b+1}^b = p_{k,1}^b, \quad p_{k,n^b+1}^t = p_{k,1}^t \quad (41)$$

5.3.3 Vertical and Diagonal Cables

Connectivity of vertical and diagonal cables is not unique. For example, it may be noticed in Figs. 6(a) and (b) that $p_{k,1}^t$ is connected to $p_{k,2}^b$ and $p_{k,3}^b$ by the vertical cables, respectively, leading to different topology. To illuminate this difference, we introduce the parameters c^v and c^d to define the connectivity of vertical and diagonal cables as follows:

- vertical cable: The connectivity of the vertical cables are defined by using an integer $c^v \in \{1, \dots, n^b\}$ as

$$P_{k,j}^V = [p_{k,j}^t, p_{k,j+c^v}^b] \quad (42)$$

where $j = 1, \dots, n^b, k = 1, \dots, n^l$, and $j + c^v = j + c^v - n^b$ if $j + c^v > n^b$.

- diagonal cable: The connectivity of the diagonal cables are defined by using an integer $c^d \in \{0, \dots, n^b - 1\}$ as

$$\begin{aligned} P_{k,j}^{D_b} &= [p_{k,j}^b, p_{k+1,j+c^d}^b] \\ P_{k,j}^{D_t} &= [p_{k,j}^t, p_{k+1,j+c^d}^b] \end{aligned} \quad (43)$$

where $j = 1, \dots, n^b, k = 1, \dots, n^l - 1$, and $j + c^d = j + c^d - n^b$ if $j + c^d > n^b$.

From a general n^l -layer tensegrity tower with n^b struts in each layer, the numbers of struts m^b , horizontal cables m^h , vertical cables m^v , saddle cables m^s and diagonal cables m^d can be written as

$$\begin{aligned} m^b &= n^l n^b, & m^h &= 2n^b, \\ m^s &= 2(n^l - 1)n^b, & m^v &= n^l n^b, \\ m^d &= 2(n^l - 1)n^b, \end{aligned} \quad (44)$$

and the number m of all members of the structure is

$$m = 6n^l n^b - 2n^b \quad (45)$$

Following the definition of connectivity of each type of members and the numbering in Eq. (37), the connectivity matrix $C \in \mathfrak{R}^{m \times n}$ can be easily constructed.

6. NUMERICAL EXAMPLES

In this section, several tensegrity towers with different numbers of layers and struts in each layer are considered as numerical examples. They are used to investigate the validity of the proposed method and to demonstrate its capacity of systematically finding self-equilibrated configurations subjected to geometrical constraints.

In the following examples, the connectivities of vertical and diagonal cables are fixed as $c^v = 1$ and $c^d = 0$ for simplicity. The $r^*(= 4)$ eigenvalues of the

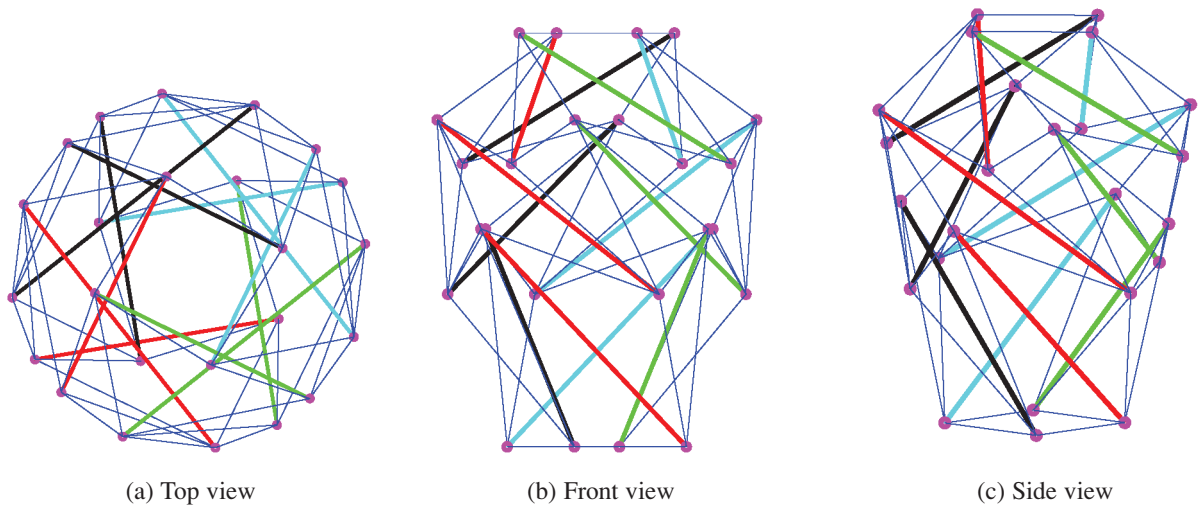


Fig. 7. A three-layer tensegrity tower with four struts in each layer.

force density matrix with the smallest values are assigned to zero in Algorithm 1, in order to obtain a non-degenerate structure in three-dimensional space. In the numerical computations, eigenvalues of the force density matrix are regarded as zero if they are less than 10^{-14} .

Table 1. Elevation of the three-layer tensegrity tower.

H_1	H_2	H_3	h_1	h_2	h_3	H
10.0	8.0	6.0	0.0	3.0	2.0	19.0

6.1 Three-layer Tensegrity Tower

Consider a tensegrity tower as shown in Fig.7 as the first example structure. It consists of three layers and four struts in each layer; i.e., $n^l = 3$ and $n^b = 4$. The structure is composed of 24 nodes and 64 members, including 12 struts, 8 horizontal cables, 12 vertical cables, 16 saddle cables and 16 diagonal cables. To simplify the computation, the saddle cables that are continuously connected to each other are classified into one group, and other types of members in the

same orbit are also classified into one group, so that the members in the same group have the same force densities. In total, there are 14 groups.

As an example, we assign the elevation of the structure as listed in Table 1. Note that the heights of every layer and the overlaps are not uniform. The total height H of the structure is 19.0.

To start Algorithm 1 in the first design stage, the initial force densities of all struts and all cables are assigned as -1 and $+1$, respectively. Algorithm 1 runs 394 iterative steps for finding the feasible force densities as listed in Table 2, where q^{b_i} , q^{h_i} , q^{v_i} , q^{s_i} and q^{d_i} denote the force densities of the groups of struts, horizontal cables, vertical cables, saddle cables and diagonal cables in layer i , respectively.

The final force density matrix \hat{E} has four zero eigenvalues and 20 positive eigenvalues. Hence, \hat{E} has rank deficiency of four, which satisfies the non-degeneracy condition of a tensegrity structure in three-dimensional space.

In the second design stage, there are up to four independent coordinates in xy -plane that can be arbitrarily specified, while the constraint of four-fold rotational symmetry is considered. Based on the

Table 2. Feasible force densities of each group of members.

q^{b_1}	q^{b_2}	q^{b_3}	q^{h_1}	q^{h_2}	q^{v_1}	q^{v_2}	q^{v_3}
-1.1656	-1.1226	-1.2366	1.2758	1.2652		0.5718	
q^{s_1}	q^{s_2}		q^{d_1}	q^{d_2}	q^{d_3}	q^{d_4}	
1.4572	1.4547		0.8484	0.8262	0.5609	0.4190	

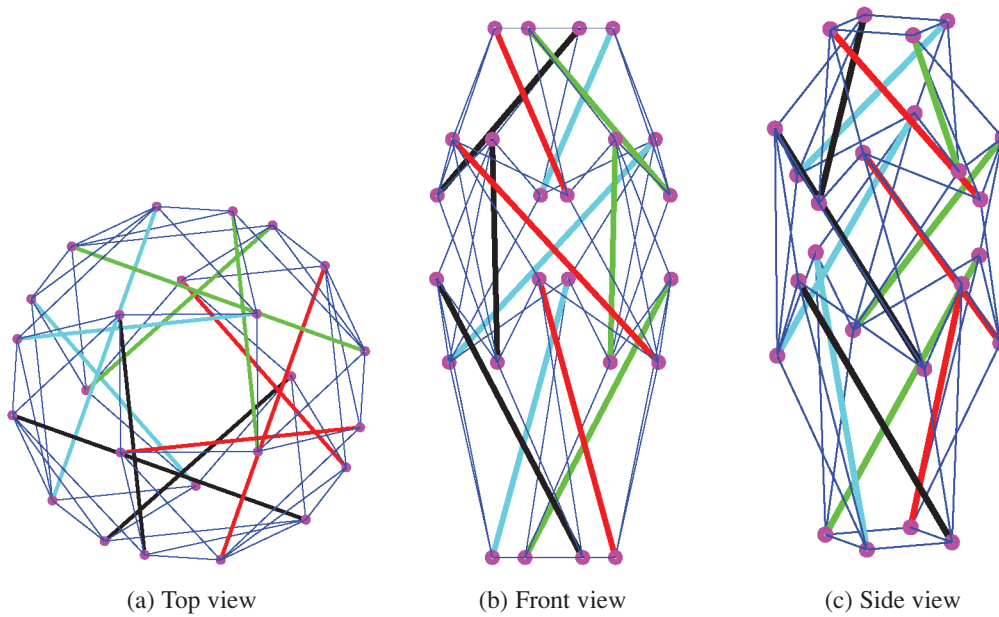


Fig. 8. New configuration of the symmetric three-layer tensegrity tower with the same force densities and coordinates in z -direction but different coordinates in xy -plane.

algorithm of consecutively specifying the independent set of nodal coordinates described in [9], the xy -coordinates of the nodes $p_{1,1}^b$ and $p_{1,1}^t$ connected by the strut $P_{1,1}^B$ in the lowest layer are selected to be specified. Note that this is not the only independent set of coordinates. If the xy -coordinates of these two nodes are specified as (10.0, 0.0) and (2.5, 4.0), then the configuration of the structure is uniquely determined as shown in Fig.7. It is easy to observe from the top view of the structure that the struts in the same layer (orbit) are rotationally symmetric by the angle $\pi/2$.

If the same nodal coordinates in xy -plane are specified to the strut $P_{2,1}^B$ in the second layer, configuration of the structure is then uniquely determined as shown in Fig.8. This new configuration of the structure becomes slightly slender compared to the configuration in Fig.7. Note that only the nodal coordinates in xy -plane have been changed in the second stage of the form-finding process. Therefore, Algorithm 1 need not be applied again to find the feasible force densities.

From the sufficient conditions for stability of tensegrity structures presented in [7], where the force density matrix should be positive semi-definite with minimum rank deficiency and rank of the geometry matrix should be six for three-dimensional structures, both of the structures considered above are guaranteed to be stable irrespective of selection of materials and level of self-stresses.

6.2 Ten-layer Tensegrity Tower

As a more complex example, a ten-layer tensegrity tower as shown in Fig.9 with four struts in each layer; i.e., $n^l = 10$ and $n^b = 4$, is considered. The structure is composed of 80 nodes and 232 members.

For simplicity, the heights and overlaps are uniformly assigned as $H_i = 10.0$ and $h_i = 2.0$ except for $h_1 = 0.0$, respectively. The total height H is 82.0. Constraint on rotational symmetry is also incorporated for this structure. The initial force densities of all struts and cables are given as -1 and $+1$, respectively.

After 511 iterative steps in Algorithm 1 for finding the feasible force densities, four independent nodal coordinates in xy -plane need to be specified for this symmetric ten-layer tensegrity tower. If the xy -coordinates of the nodes $p_{2,1}^b$ and $p_{2,1}^t$ connected by the strut $P_{2,1}^B$ in layer 2 are specified as (10.0, 0.0) and (2.5, 4.0), its configuration is obtained as shown in Fig.9, where the top view and the front view have been drawn in different scales.

The force density matrix of this structure is positive semi-definite with four zero eigenvalues. So the necessary condition for a non-degenerate tensegrity structure is satisfied. Moreover, from the sufficient conditions for stability of tensegrity structures in [7], this structure is also guaranteed to be stable.

In all of the examples presented in this section, the errors ϵ calculated by Eq. (32) are within 10^{-10} . Hence, it can be concluded that the final configurations in

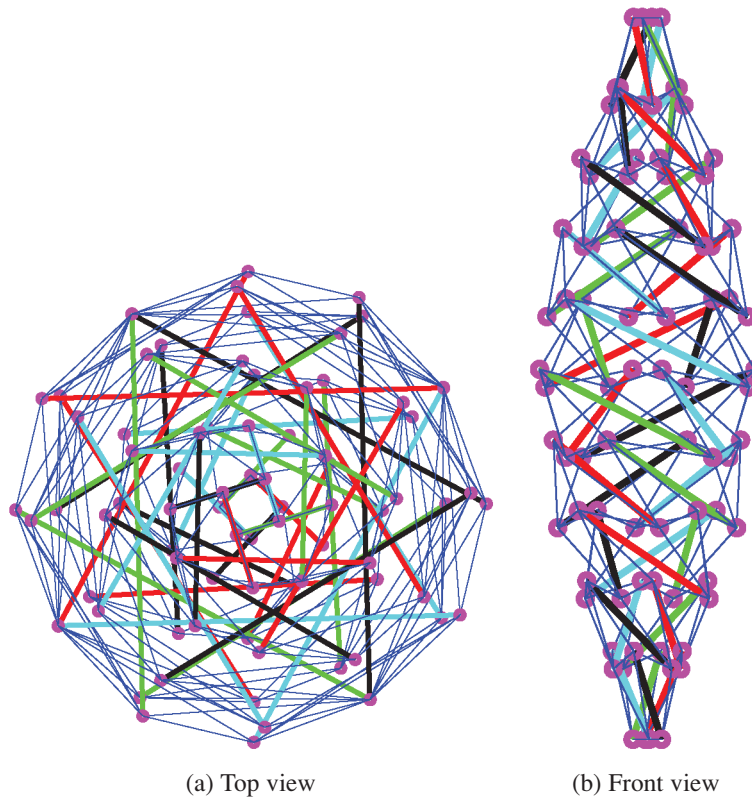


Fig. 9. A ten-layer tensegrity tower with four struts in each layer.

these examples derived by the proposed method satisfy the self-equilibrium equations precisely.

By modifying the values of the initial force densities and the independent nodal coordinates, more new and interesting configurations can be systematically found.

7. DISCUSSIONS AND CONCLUSIONS

A substantial improvement to the adaptive force density method has been made for finding self-equilibrated configurations of tensegrity structures to satisfy the specified geometrical constraints. The geometrical constraints are formulated in linear forms with respect to force densities and nodal coordinates so as to be incorporated into the adaptive force density method. The proposed form-finding process is divided into two design stages: (1) finding the feasible force densities that satisfy the non-degeneracy condition for tensegrity structures and linear constraints with respect to them, and (2) determining the self-equilibrated configuration that satisfies the self-equilibrium equations as well as linear constraints with respect to nodal coordinates.

In the proposed form-finding process, the following parameters are necessary as inputs:

(1) topology;

(2) geometrical constraints;

(3) an initial set of force densities;

(4) an independent set of nodal coordinates.

Tensegrity towers were used as numerical examples to illustrate the capability of the proposed method for finding proper configurations subjected to given geometrical constraints. Moreover, based on the detailed description of the geometry of tensegrity towers, designers can avoid the tedious job of modelling their topology, so as to concentrate on design aspect according to their preferences.

Besides tensegrity towers, the proposed method can also be applied to other types of tensegrity structures subjected to geometrical constraints. And any other kinds of geometrical constraints can also be incorporated into the proposed form-finding process as long as they can be formulated in linear forms with respect to force densities and nodal coordinates, similar to the example constraints on rotational symmetry and elevation considered in this study.

Positive definiteness of the tangent stiffness matrix of a structure is widely adopted as the stability of a structure in the field of structural engineering. The tangent stiffness matrix is the sum of the linear and geometrical stiffness matrices. The linear stiffness

matrix is always positive semi-definite, while positive definiteness of the geometrical stiffness matrix depends on the distribution of self-stresses. Because the geometrical stiffness matrix is composed of the force density matrix as formulated in [5], ensuring a positive semi-definite force density matrix will tend to lead to a stable structure, although some other conditions should also be satisfied to be the sufficient conditions [7]. Hence, assignment of zero to smallest eigenvalues to get rid of the negative eigenvalues of the force density matrix in Algorithm 1 will naturally lead to positive semi-definite force density matrix and then to a stable structure.

By applying the proposed method, we can have direct and exact controls over some geometrical properties, but it is unlikely to control all aspects of a tensegrity structure, which is considered as one of the common shortcomings of the family of force density method. For example, it is not easy to exactly assign the lengths of all members, because these constraints are not easy to be written in linear terms with respect to force densities and nodal coordinates.

ACKNOWLEDGEMENT

The first author acknowledges support from The Kyoto University Foundation and thanks Dr. Simon D. Guest of Cambridge University for his helpful suggestions in writing the paper.

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