Form-finding of Tensegrity Structures Subjected to Geometrical Constraints

J.Y. Zhang[1] and M. Ohsaki[2]
[1,2] *Department of Architecture & Architectural Engineering, Kyoto University, Kyoto-Daigaku Katsura, Nishikyo, Kyoto 615-8540, Japan*
[1]is.zhang@archi.kyoto-u.ac.jp, [2]ohsaki@archi.kyoto-u.ac.jp

Abstract

Geometrical constraints on symmetry and elevation of a tensegrity structure are formulated as linear equations with respect to force densities and nodal coordinates. These constraints are incorporated into the adaptive force density method for finding the non-degenerate and self-equilibrated configuration of the structure. The form-finding process is divided into two design stages: (1) finding the feasible force densities that satisfy the non-degeneracy condition, and (2) unique determination of the configuration (nodal location) satisfying the equilibrium conditions. The proposed method is demonstrated to be powerful for systematically searching new configurations subjected to the geometrical constraints.

Key words: Tensegrity Structure; Symmetry; Form-finding; Geometrical Constraint; Force Density Method.

1. Introduction

In the shape design of a tensegrity structure, geometrical properties are usually concerned by the designers. For example, symmetric configuration is generally regarded as more beautiful and preferable than the asymmetric one. Due to the interaction between the self-stresses in the members and the configuration, geometrical properties of a tensegrity structure are not easy to be exactly controlled. This motivates us to present a general methodology to find the proper configuration, the so-called *form-finding* problem, of the structure subjected to the specified geometrical constraints.

The word *tensegrity* – contraction of tension and integrity, is introduced by Fuller [1]. The structure consisting of six nodes and nine members as shown in Fig.1 is the simplest three-dimensional tensegrity structure. The members in thick lines are struts which contain only compression and the ones in thin lines are cables containing only tension. Note that the struts do not contact with any other strut. This is

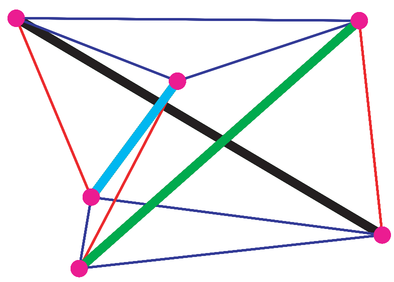


Fig. 1. The simplest three-dimensional tensegrity structure.

the original and and strict definition of the tensegrity structures. Thus, every node of the structure should be physically connected with only one strut.

Tensegrity structure belongs to the Type III pin-jointed structure according to the following classification for general pin-jointed structures based on the statical stability investigation [2]:

- Type I: *truss*, which contains no self-stress.
- Type II: *tensile structure*, the members of which are in tension.
- Type III: *tensegrity structure*, which has both tensile members (cables) and compressional members (struts).

The force density method [3], which transforms the non-linear self-equilibrium equations with respect to the nodal coordinates into a set of linear equations by introducing the concept of force density, force-to-length ratio, can be easily applied to the form-finding problems of tensile structures because the equilibrium matrix is positive definite in these cases.

The difficulty in application of the basic idea of force density method to tensegrity structure is that the structure may degenerate to a plane, a line or even a point, because of the singularity of the associated equilibrium matrix and the existence of both tensile and compressional members. In order to find the feasible force densities that satisfy the non-degeneracy condition of a tensegrity structure, an analytical method [4] and a numerical method [5] were presented.

When the structure is relatively simple, the analytical method can find the relationship among the force densities satisfying the non-degeneracy condition so as to provide a deeper insight of the mechanical properties of the structure. On the other hand, when the structure is relatively complex and has many nodes and members, the analytical method might not be efficient enough and the relationship among the unknown parameters may turn out to be less meaningful because of the large number of them. In this case, the numerical method, called *adaptive force density method*, can be a more powerful tool for systematically searching new configurations.

However, both of the above-mentioned methods cannot have exact control over the geometrical properties of a tensegrity structure, e.g, asymmetric configuration may be obtained although the symmetric one is expected. This is because that the geometrical constraints have not yet been explicitly included in the formulations.

To enable designers to directly control the configuration, the adaptive force density method is extended and divided into two design stages in this study. The geometrical constraints regarding symmetry and elevation are formulated in linear forms with respect to force densities and nodal coordinates. The constraints with respect to force densities are incorporated into the first design stage of the method to find the feasible set of force densities satisfying the non-degeneracy condition. And the constraints with respect to nodal coordinates are incorporated in the second design stage to uniquely determine the preferred configuration of the structure satisfying self-equilibrium conditions.

The paper following this introduction is organized as follows: Sections 2 briefly gives the self-equilibrium equations and the non-degeneracy condition of a tensegrity structure. Section 3 presents the formulations of the geometrical constraints on the structure, including symmetry and elevation. The geometrical constraints are incorporated with the adaptive force density method to find the proper configuration of the structure in Section 4. Section 5 gives a detailed description on the geometrical properties of tensegrity towers, which are used as examples to illustrate the validity and capability of the proposed method in Section 6. Section 7 discusses the advantages and disadvantages of the proposed method and then concludes the study.

2. Self-equilibrium Equations and Non-degeneracy Condition

The topology of a tensegrity structure composed of the members that transmit only axial forces can be described in terms of the connectivity matrix $\mathbf{C} \in \Re^{m\times n}$, where m and n are the numbers of members and nodes, respectively. If member k is connected to nodes i and j ($i < j$), then the ith and jth elements of the kth row of $\mathbf{C}$ are equal to $+1$ and -1, respectively, while other elements in the row are zero. Note that $\mathbf{C}$ is a constant matrix when the topology of the structure is given. This way, the topology of a tensegrity structure can be defined as a directed graph [6].

Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ ($\in \Re^n$) denote the vectors of x-, y- and z-coordinates, respectively, of the nodes. Denote the force density q_k of member k as the force s_k to length l_k ratio; i.e., $q_k = s_k/l_k$. Since a strut and a cable can transmit only compression and tension forces, respectively, we have $q_k < 0$ for a strut and $q_k > 0$ for a cable. The force density vector consisting of q_k ($k = 1, \ldots, m$) is denoted by $\mathbf{q} \in \Re^m$.

When there is no external load applied, the structure is in a state of self-equilibrium, and the self-equilibrium equations with respect to the nodal coordinate vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ in each direction can be respectivly written as [5]

$$\mathbf{E}\mathbf{x} = \mathbf{0} \tag{1.1}$$

$$\mathbf{E}\mathbf{y} = \mathbf{0} \tag{1.2}$$

$$\mathbf{E}\mathbf{z} = \mathbf{0} \tag{1.3}$$

where $\mathbf{E} \in \Re^{n\times n}$ can be formulated by using the connectivity matrix $\mathbf{C}$ and the diagonal version of the force density vector as

$$\mathbf{E} = \mathbf{C}^\top \mathrm{diag}(\mathbf{q})\mathbf{C} \tag{2}$$

From the definition of the force density, we may know that the self-equilibrium equations (1.1)–(1.3) are non-linear with respect to the nodal coordinates. In the case that the force densities are known or given, $\mathbf{E}$ can be regarded as a constant matrix, and the originally non-linear equations are transformed into a set of linear equations with respect to the nodal coordinate vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, respectively. This is the basic idea of the force density method.

Since $\mathbf{E}$ in the self-equilibrium equations represents the equilibrium with respect to the nodal coordinates of the structure, it is called *equilibrium matrix* in the study. Note that $\mathbf{E}$ may also be called stress matrix [7] or force density matrix [4] in some other studies.

From the definition in Eq. (2), $\mathbf{E}$ is always square and symmetric, furthermore, it has rank deficiency of at least one, because the sum of the components in each row or column is 0.

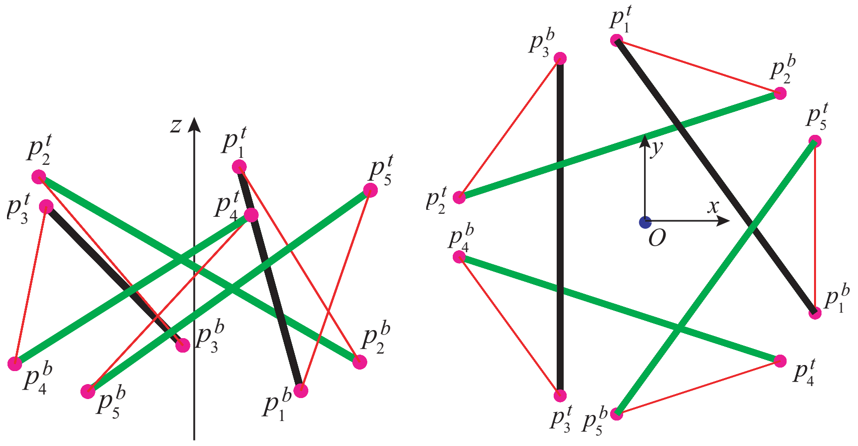
Let h denote the rank deficiency of $\mathbf{E}$ as

$$h = n - \mathrm{rank}(\mathbf{E}) \tag{3}$$

The solution space of each linear self-equilibrium equations in (1) can then be described as a linear combination of h independent vectors. As discussed in [5] based on the solution spaces, h of a non-degenerate d-dimensional (d=2 or 3) tensegrity structure should be equal to or larger than the necessary rank deficiency $h^\star$ as

$$h \geq h^\star = d + 1 \tag{4}$$

When Eq. (4) is not satisfied, the structure in the d-dimensional space will degenerate into a space with lower dimen-



(a) perspective view (b) plan

Fig. 2. An example of rotational symmetry of five struts.

sions, such as a plane ($h = 3$), a line ($h = 2$) or a point ($h = 1$). Hence, the problem of finding the configuration of a non-degenerate three-dimensional tensegrity structure turns out to be a process of finding a feasible set of force densities that satisfies Eq. (4) with $d = 3$.

It should be noted that this is only the necessary but not sufficient condition for a non-degenerate tensegrity structure; i.e., the structure satisfying Eq. (4) may still be degenerate due to the inappropriately specified nodal coordinates.

3. Geometrical Constraints

In this section, geometrical constraints on the symmetry and elevation of a tensegrity structure are separately formulated. The rotational symmetry about the z-axis of the structure is formulated as linear equations with respect to the force density vector and xy-coordinates of the nodes. These two conditions should be satisfied in the first and the second design stages, respectively. The constraints on the elevation of the structure are written in a linear form with respect to the force density vector and incorporated in the first design stage.

3.1 Symmetry Constraints with Respect to Force Densities

n^b-fold symmetry of a structure refers to the fact that a member among a set of n^b members can be moved to any other member in the set by the rotation about the z-axis by $2i\pi/n^b$, where i is an appropriate integer value. These n^b members are said to be in the same *orbit* and have the same lengths and forces, and the same force densities accordingly. Note that there are usually more than one orbit of struts in a symmetric structure.

For example, the five struts shown in thick lines in Fig.2 belong to the same orbit. Any one of the strut can be moved to another by rotation about the z-axis by $2i\pi/5$ ($i \in \{1, 2, 3, 4\}$). The cables in thin lines belong to a different orbit.

Since the force densities of the members in the same orbit of a symmetric structure have the same value, the symmetry of the structure with respect to the force density vector can be written as

$$\mathbf{Fq} = \mathbf{0} \tag{5}$$

where there are only two non-zero elements $+1$ and -1 in each row of $\mathbf{F}$. For example, if members i and j ($i < j$) are in the same orbit, there must be one row k of $\mathbf{F}$ consisting of $+1$ and -1 at ith and jth elements, respectively, and the the remaining elements in the row are 0 as

$$\mathbf{F}_{(k,p)} = \begin{cases} 1 & \text{for } p = i \\ -1 & \text{for } p = j \\ 0 & \text{for other cases} \end{cases} \tag{6}$$

The constraints (5) on the force density vector will be incorporated in the first design stage of the proposed method in the next section.

3.2 Symmetry Constraints with Respect to Nodal Coordinates

Since every node of a tensegrity structure must be connected to only one strut in the strict definition of tensegrity, it is sufficient to consider only the struts of the structure while describing the symmetry properties. Moreover, since the struts in different orbits are geometrically independent in view of symmetry, the symmetry properties of a structure can be formulated only for the struts in one orbit, and then simply extended to the whole structure.

Consider a symmetric tensegrity structure with n^l orbits of struts and n^b struts in each orbit. Denote the higher and lower nodes of the strut i by p_i^t and p_i^b, respectively. And denote the x- and y-coordinates of the nodes p_i^t and p_i^b by the vectors $\mathbf{x}_i^t = (x_i^t, y_i^t)^\top$ and $\mathbf{x}_i^b = (x_i^b, y_i^b)^\top$, respectively.

The directed member vector $\mathbf{d}_i \in \Re^2$ $(i = 1, \dots, n^b)$ of the struts on xy-plane is defined as

$$\mathbf{d}_i = \mathbf{x}_i^b - \mathbf{x}_i^t \tag{7}$$

which corresponds to the edges of a directed graph [6]. For example, Fig.3 shows the directed graph of the five struts in Fig.2.

$\mathbf{d}_i$ in the kth orbit of struts of the structure are combined to $\mathbf{d}^k \in \Re^{2n^b}$ as

$$\mathbf{d}^k = (\mathbf{d}_1^\top, \dots, \mathbf{d}_{n^b}^\top)^\top \tag{8}$$

And $\mathbf{x}_i^b$ and $\mathbf{x}_i^t$ in the orbit are combined to

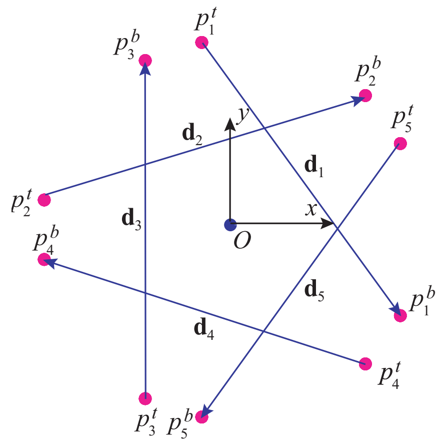


Fig. 3. Directed graph of the five struts in Fig.2.

$\mathbf{x}^k \in \Re^{4n^b}$ as

$$\mathbf{x}^k = \left((\mathbf{x}_1^b)^\top, \dots, (\mathbf{x}_{n^b}^b)^\top, (\mathbf{x}_1^t)^\top, \dots, (\mathbf{x}_{n^b}^t)^\top\right)^\top \tag{9}$$

The relationship between $\mathbf{d}^k$ and $\mathbf{x}^k$ in orbit k can be written as follows by a matrix $\mathbf{T}^k \in \Re^{2n^b \times 4n^b}$:

$$\mathbf{d}^k = \mathbf{T}^k \mathbf{x}^k \tag{10}$$

where the matrix $\mathbf{T}^k$ is constructed by the $2n^b$-by-$2n^b$ identity matrix $\mathbf{I}^{2n^b}$ in a form as

$$\mathbf{T}^k = \begin{pmatrix} \mathbf{I}^{2n^b} & -\mathbf{I}^{2n^b} \end{pmatrix} \tag{11}$$

The symmetry properties of a structure can be easily described by the group representation theory. The n^b struts in an orbit constitute a cyclic group of order n^b. Details of cyclic group and its representation theory can be found in many mathematics or chemistry textbooks, e.g., [8]. However, knowledge on group representation theory is not necessary here, because the two-dimensional E_1 representation matrix $\mathbf{R}_i$ of the cyclic group defined as follows is identical to the transformation matrix of counter-clockwise rotation about the z-axis by $2(i-1)\pi/n^b$, which might be more familiar to the designers:

$$\mathbf{R}_i = \begin{pmatrix} C_i & -S_i \\ S_i & C_i \end{pmatrix} \quad (12)$$

where $C_i = \cos(2(i-1)\pi/n^b)$ and $S_i = \sin(2(i-1)\pi/n^b)$.

If $\mathbf{d}_1$ coincides with $\mathbf{d}_i$ by the counter-clockwise rotation about z-axis by the angle $2(i-1)\pi/n^b$, the following relation holds:

$$\mathbf{d}_i = \mathbf{R}_i \mathbf{d}_1 \quad (13)$$

Eq. (13) can be rewritten with respect to $\mathbf{d}^k$ as follows by using the 2-by-2 identity matrix $\mathbf{I}^2$:

$$\begin{pmatrix} \mathbf{R}_i & \dots & -\mathbf{I}^2 & \dots \end{pmatrix} \mathbf{d}^k = \mathbf{0} \quad (14)$$

Combining all the relations of $\mathbf{d}_i$ ($i \neq 1$) and $\mathbf{d}_1$ similar to Eq. (14) by using the matrix $\mathbf{S}^k \in \Re^{2(n^b-1)\times 2n^b}$, we obtain

$$\mathbf{S}^k \mathbf{d}^k = \mathbf{0} \quad (15)$$

By substituting Eq. (10) into Eq. (15) and letting $\bar{\mathbf{S}} = \mathbf{S}^k \mathbf{T}^k \in \Re^{2(n^b-1)\times 4n^b}$, the rotational symmetry of the struts in orbit k can be expressed in a linear form with respect to the nodal coordinates in xy-plane as

$$\bar{\mathbf{S}} \mathbf{x}^k = \mathbf{0} \quad (16)$$

Because the rotational symmetry of the struts in different orbits can be formulated independently, the rotational symmetry of the whole structure in terms of the generalized coordinate vector $\mathbf{X} = \left((\mathbf{x}^1)^\top, \dots, (\mathbf{x}^{n^l})^\top\right)^\top \in \Re^{2n}$ in xy-plane can then be written as

$$\mathbf{S}\mathbf{X} = \mathbf{0} \quad (17)$$

where $\mathbf{S} \in \Re^{2n^l(n^b-1)\times 4n^l n^b}$ is the tensor product of an n^l-by-n^l identity matrix $\mathbf{I}^{n^l}$ and the matrix $\bar{\mathbf{S}}$; i.e., $\mathbf{S} = \mathbf{I}^{n^l} \otimes \bar{\mathbf{S}}$.

This way, the symmetry properties of the whole structure can be formulated as a set of linear equations with respect to the generalized nodal coordinate vector $\mathbf{X}$ in xy-plane. The constraints Eq. (17) will be incorporated in the second design stage to ensure a rotationally symmetry of the structure.

3.3 Constraints on Elevation

Suppose that the elevation of the structure is assigned by the designer. So z-coordinates of all the nodes are determined.

Since the following relation always holds for any vectors $\mathbf{q} \in \Re^m$, $\mathbf{z} \in \Re^n$ and matrix $\mathbf{C} \in \Re^{m\times n}$:

$$\mathrm{diag}(\mathbf{q})\mathbf{C}\mathbf{z} = \mathrm{diag}(\mathbf{C}\mathbf{z})\mathbf{q} \quad (18)$$

the self-equilibrium equation (1.3) in z-direction can be rewritten with respect to $\mathbf{q}$ as

$$\mathbf{C}^\top \mathrm{diag}(\mathbf{C}\mathbf{z})\mathbf{q} = \mathbf{0} \quad (19)$$

By letting $\mathbf{H} = \mathbf{C}^\top \mathrm{diag}(\mathbf{C}\mathbf{z})$, the geometrical constraints on the elevation of a tensegrity structure with respect to $\mathbf{q}$ can then be written as

$$\mathbf{H}\mathbf{q} = \mathbf{0} \quad (20)$$

which is incorporated in the first design stage in the next section for finding the feasible set of force densities.

4. Form-finding Process

This section demonstrates how the geometrical constraints on symmetry and elevation of a tensegrity structure is incorporated into the adaptive force density method.

4.1 Constraints on the Force Density Vector

By assembling the columns of the equilibrium matrix $\mathbf{E}$ to the vector $\mathbf{g} \in \Re^{n^2}$, the relation between $\mathbf{g}$ and the force density vector $\mathbf{q}$ can be written as follows by using a constant matrix $\mathbf{B} \in \Re^{n^2 \times m}$ [5]

$$\mathbf{B}\mathbf{q} = \mathbf{g} \tag{21}$$

where each row of $\mathbf{B}$ has only two nonzero elements $+1$ and -1.

So far, we have three linear equations (5), (20) and (21) with respect to $\mathbf{q}$, which are related to symmetry, elevation and the equilibrium matrix, respectively. Eq. (21) have to be strictly satisfied, because it is derived from the definition of $\mathbf{E}$, as the geometrical constraints (5) and (20) should be satisfied in view of designer's preference.

Combine Eqs. (5) and (20) as

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{H} \end{pmatrix} \mathbf{q} = \mathbf{0} \tag{22}$$

Since the matrix in the linear equation (22) with respect to $\mathbf{q}$ is usually rank deficient, the solution of (22) can be written as

$$\mathbf{q} = \mathbf{\Psi}\boldsymbol{\alpha} \tag{23}$$

where $\boldsymbol{\alpha}$ is a coefficient vector, and the columns of the matrix $\mathbf{\Psi}$ span the solution space of Eq. (22). Note that $\mathbf{\Psi}$ is also a constant matrix when the constraints are given. Since $\mathbf{E}$ has to satisfy the non-degeneracy condition (4) and $\mathbf{q}$ is related to $\mathbf{E}$ through Eq. (21), the coefficient vector $\boldsymbol{\alpha}$ cannot be selected arbitrarily.

4.2 Feasible Force Density Vector (First Design Stage)

As the first design stage of the form-finding process based on the adaptive force density method, an iterative algorithm is presented to find the feasible force densities satisfying the non-degeneracy condition (4) and the linear constraints (21) and (22).

Suppose that we have obtained the force density vector $\mathbf{q}^i$, the corresponding equilibrium matrix $\mathbf{E}^i$ of which has the necessary rank deficiency, at the ith step of the iterative algorithm. Substituting Eq. (23) into Eq. (21), we have

$$\mathbf{g}^i = \mathbf{B}\mathbf{\Psi}^i\boldsymbol{\alpha}^i \tag{24}$$

Since $\mathbf{B}\mathbf{\Psi}^i$ in Eq. (24) is usually full-rank and not square, the coefficient vector can be computed as follows by using the least square method

$$\boldsymbol{\alpha}^i = (\mathbf{B}\mathbf{\Psi}^i)^- \mathbf{g}^i \tag{25}$$

where $(\ )^-$ denotes the generalized inverse matrix. The force density vector $\mathbf{q}^i$ can be updated to $\mathbf{q}^{i+1}$ by Eq. (23) as

$$\mathbf{q}^{i+1} = \mathbf{\Psi}^i(\mathbf{B}\mathbf{\Psi}^i)^- \mathbf{g}^i \tag{26}$$

Note that $\mathbf{q}^{i+1}$ may not be equal to $\mathbf{q}^i$, so the new equilibrium matrix $\mathbf{E}^{i+1}$ corresponding to $\mathbf{q}^{i+1}$ may not have the necessary rank deficiency $h^\star$ and has to be recomputed based on the following eigenvalue analysis and spectral decomposition.

The symmetric equilibrium matrix $\mathbf{E}^i$ can be decomposed as [9]

$$\mathbf{E}^i = \mathbf{\Phi}^i\mathbf{\Lambda}^i(\mathbf{\Phi}^i)^\top \tag{27}$$

where the columns of $\mathbf{\Phi}^i$ are the eigenvectors of $\mathbf{E}^i$, and the diagonal elements $\{\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i\}$ of the diagonal matrix $\mathbf{\Lambda}^i$ are the eigenvalues of $\mathbf{E}^i$. The number of nonzero eigenvalues of $\mathbf{E}^i$ is equal to its rank.

Let r^i denote the number of non-positive eigenvalues of $\mathbf{E}^i$. There are two cases discussed in [5]: (a) $r^i \le h^\star$ and (b) $r^i > h^\star$. From our numerical experience, $r^i \le h^\star$ is usually the case, so we focus on only this case.

In order to make the equilibrium matrix have the necessary rank deficiency $h^\star$, we can assign the $h^\star$ eigenvalues with the smallest (absolute) values to zero. The diagonal matrix $\mathbf{\Lambda}$ is updated as $\bar{\mathbf{\Lambda}}^i$ with the new eigenvalues, and the equilibrium matrix is updated as $\bar{\mathbf{E}}^i$:

$$\bar{\mathbf{E}}^i = \mathbf{\Phi}^i \bar{\mathbf{\Lambda}}^i (\mathbf{\Phi}^i)^\top \tag{28}$$

Obviously, $\bar{\mathbf{E}}^i$ has the necessary rank deficiency $h^\star$.

By iteratively applying Eqs. (28) and (26), we can adaptively find the feasible force density vector $\hat{\mathbf{q}}$, which can be summarized as follows:

Algorithm 1: Feasible force density vector

Step 0: Give an initial $\mathbf{q}^0$ to obtain $\mathbf{E}^0$ by Eq. (2). Set $i := 0$.

Step 1: Assign 0 to the $h^\star$ smallest (absolute) eigenvalues of $\mathbf{E}^i$ and reconstruct $\bar{\mathbf{E}}^i$ by Eq. (28).

Step 2: Obtain $\mathbf{g}^{i+1}$, calculate $\mathbf{q}^{i+1}$ from Eq. (26) and update $\mathbf{E}^{i+1}$ by Eq. (2).

Step 3: If Eq. (4) holds, then let $\hat{\mathbf{q}} = \mathbf{q}^{i+1}$, compute $\hat{\mathbf{E}}$ and terminate the algorithm; otherwise, set $i \leftarrow i+1$ and return to Step 1.

4.3 Determination of Configuration (Second Design Stage)

As the elevation of the structure is given by the designers, the z-coordinates of the nodes are then determined. And only the generalized nodal coordinates $\mathbf{X}$ in xy-plane, which is subjected to the geometrical constraint (17) on rotational symmetry and the self-equilibrium equations (1.1) and (1.2), is needed to be determined to obtain the configuration of the structure. The constraints are combined to the following linear equation with respect to $\mathbf{X}$

$$\begin{pmatrix} \mathbf{E} \otimes \mathbf{I}^2 \\ \mathbf{S} \end{pmatrix} \mathbf{X} = \mathbf{0} \tag{29}$$

The solution of Eq. (29) can be written as

$$\mathbf{X} = \mathbf{\Omega} \boldsymbol{\beta} \tag{30}$$

where $\boldsymbol{\beta}$ is the coefficient vector. If $\boldsymbol{\beta}$ is a r^c-dimensional vector, then the designers have freedom of specifying r^c independent xy-coordinates denoted by $\mathbf{X}^c \in \Re^{r^c}$.

The rows of $\mathbf{\Omega}$ corresponding to $\mathbf{X}^c$ are assembled to $\mathbf{\Omega}^c \in \Re^{r^c \times r^c}$, and $\mathbf{X}$ can be computed as

$$\mathbf{X} = \mathbf{\Omega} (\mathbf{\Omega}^c)^{-1} \mathbf{X}^c \tag{31}$$

Together with the z-coordinates assigned by the designers, configuration of the structure in terms of nodal coordinates can have been uniquely determined.

There are many ways to specify the independent set of nodal coordinates $\mathbf{X}^c$. A method for identifying the dependent nodal coordinates based on the Reduced Row-Echelon Form (RREF) of $\mathbf{\Omega}^\top$ was presented in [10], where the designers can specify the nodal coordinates consecutively according to their preference.

The independent nodal coordinates can also be determined in only one step based on the RREF of $\mathbf{\Omega}^\top$. A simple example of this approach can be found in [5].

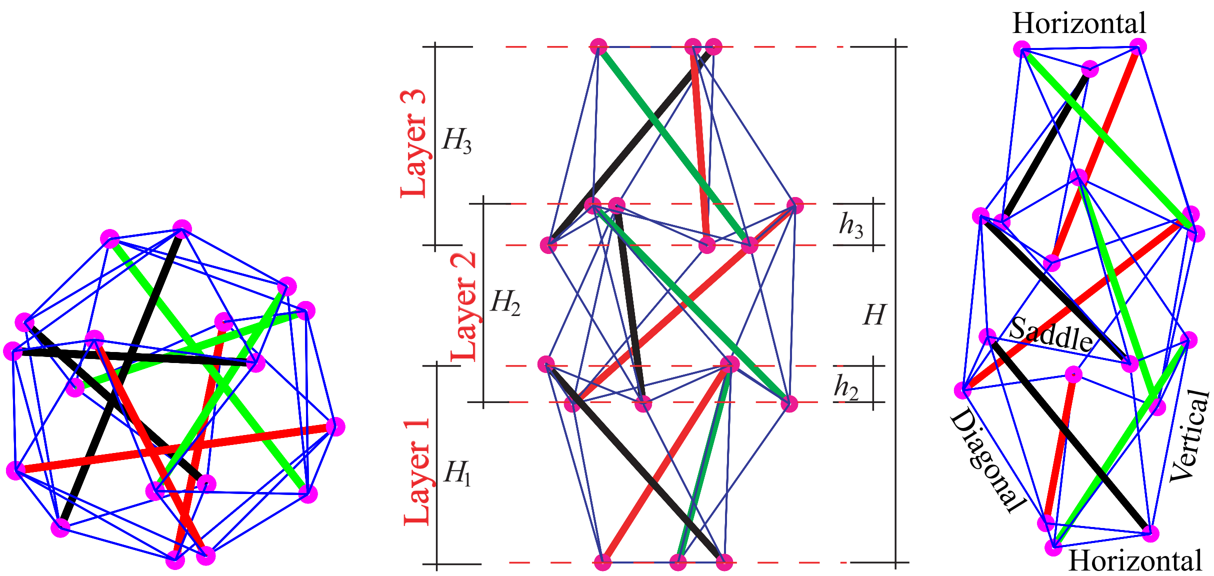
4.4 Adaptive Force Density Method with Geometrical Constraints

The process of finding the configuration of a tensegrity structure with constraints on symmetry as well as elevation can be divided into two design stages as follows:

Algorithm 2 – Form-finding Process

First Stage: *Feasible Force Density Vector*

(1) Specify the topology.
(2) Formulate the geometrical constraints with respect to the force density vector.
(3) Assign an initial force density vector.



(a) Top view (b) Side view (c) Perspective view

Fig. 4. A symmetric three-layer tensegrity tower with three struts in each layer.

(4) Find the feasible force density vector by Algorithm 1.

Second Stage: *Determination of Configuration*

(5) Formulate the geometrical constraints with respect to the nodal coordinates.

(6) Specify an independent set of nodal coordinates to uniquely determine the configuration.

As will be demonstrated in the numerical examples, designers can control the configuration of a tensegrity structure by changing the values of the parameters in Steps (1), (2), (3) and (6). Symmetry of the structure is ensured by the constraints in Steps (2) and (5).

5. Tensegrity Towers

Tensegrity tower, as shown in Fig.4, is a special kind of tensegrity structures. It has one or more layers and at least three struts in each layer. The needle tower invented and built by Kenneth Snelson in Bryant Park in New York may be one of the best-known tensegrity towers in the world.

In this section, we give a detailed description of the geometrical properties and topology of the tensegrity towers, and use them as examples in the next section to demonstrate the process of finding the desired configurations satisfying the geometrical constraints.

5.1 Configuration

Suppose that a tensegrity tower has n^l layers (orbits of struts) and n^b struts in each layer. The struts in each layer belong to the same orbit.

The nodes that have the same z-coordinate are said to be in the same plane. Thus, each layer have two different planes – the bottom and the top planes.

Since no strut physically contacts any other strut, the number n of nodes of a tensegrity tower is

$$n = 2n^l n^b \qquad (32)$$

A similar classification to the one in [11] is adopted – the cables of the tensegrity towers are classified into the following four types as shown in Fig.4.(c) based on the connectivity:

- *Horizontal cables* that connect the nodes in the same plane. They can only exist in the bottom plane of the lowest layer and the top plane of the highest layer.
- *Vertical cables* that are connected by the nodes in the top and bottom planes of the same layer.
- *Saddle cables* that connect the nodes in different planes of the adjacent layers, e.g., the top plane of layer k and the bottom plane of layer $k+1$.
- *Diagonal cables* that connect the nodes in the same top (or bottom) planes of the adjacent layers, e.g., the top (or bottom) plane of layer k and the top (or bottom) plane of the layer $k+1$.

5.2 Elevation

Denote the height of the ith layer by H_i ($i = 1, \ldots, n^l$), and the overlap between two adjacent layers i and $i-1$ by h_i ($i = 1, \ldots, n^l$) where $h_1 = 0$. Designers are free to design the elevation of the structure by assigning H_i and h_i. The total height H of the structure can be computed by

$$H = \sum_{i=1}^{n^l} (H_i - h_i) \tag{33}$$

The z-coordinates z_k^t of the nodes in the top plane of layer k ($k = 1, \ldots, n^l$) can be determined as

$$z_k^t = \sum_{i=1}^{k} (H_i - h_i) \tag{34}$$

And the coordinates z_k^b of the nodes in the bottom plane of layer k can be computed by

$$z_k^b = z_k^t - H_k \tag{35}$$

This way, the vector $\mathbf{z}$ of the z-coordinates of the tensegrity tower can be determined.

5.3 Topology

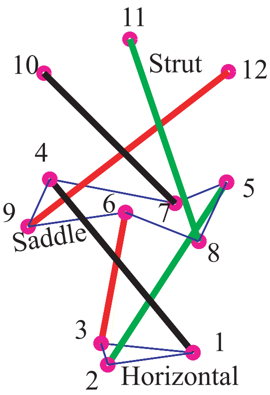


Fig. 5. An example of connectivity of struts, horizontal cables and saddle cables.

In order to formulate the connectivity matrix $\mathbf{C}$ in a simple manner for a tensegrity tower with any number of layers ($n^l \geq 1$) and any number of struts ($n^b \geq 3$) in each layer, topology of a general tensegrity tower is defined in this subsection.

The nodes in the bottom and top planes of layer k are labelled by $p_{k,j}^b$ and $p_{k,j}^t$, respectively, as

$$\begin{aligned} p_{k,j}^b &= 2(k-1)n^b + j \\ p_{k,j}^t &= (2k-1)n^b + j \end{aligned} \quad (j = 1, \ldots, n^b) \tag{36}$$

5.3.1 Struts

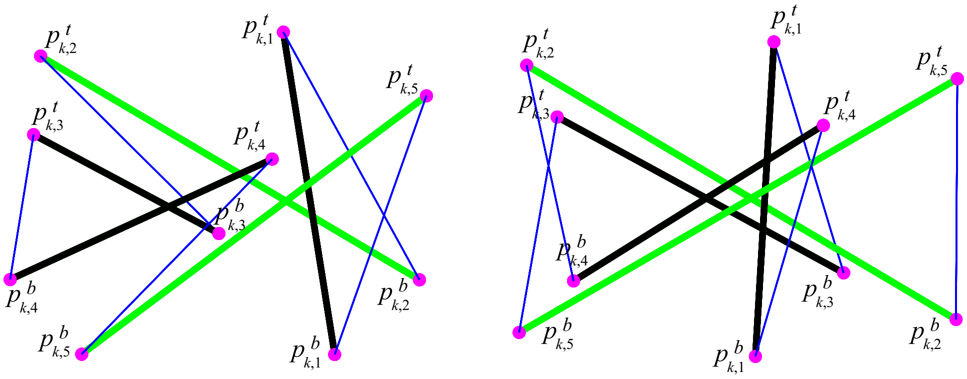
The jth strut $P_{k,j}^B$ in layer k is connected by nodes $p_{k,j}^b$ and $p_{k,j}^t$ in different planes:

$$P_{k,j}^B = [p_{k,j}^b, p_{k,j}^t] \tag{37}$$

where $[i, j]$ indicates that nodes i and j are connected to construct a member. A simple example with $n^b = 3$ is illustrated in Fig.5, where the vertical and diagonal cables are removed for clarity.

5.3.2 Horizontal and Saddle Cables

If the tangent stiffness matrix, after constraining the rigid-body motions, is positive definite, the structure is said to be *prestress stable*. When the equilibrium matrix,



(a) $c^v = 1$ (b) $c^v = 2$

Fig. 6. An example of connectivity of vertical cables in layer k.

or the geometrical stiffness matrix, is positive semi-definite, the structure is guaranteed to be prestress stable irrespective of the selection of materials and level of self-stresses, so it is said to be *super stable* in this case [7]. Apparently, super stability is usually preferred in the design of a tensegrity structure.

For the one-layer tensegrity towers ($n^l = 1$) as shown in Fig.1, it has been proved that the structure is not super stable if the horizontal cables are not connected with the adjacent nodes [7]. This is not always true for the prestress stability, which depends on the interaction of the mechanisms and the geometrical stiffness matrix [12].

In order to avoid the risk of generating an unstable structure in view of either super or prestress stability, the horizontal and saddle cables are assumed to connect the adjacent nodes as shown in Fig.5. Thus the connectivity of horizontal and vertical cables are given as follows:

- *horizontal cables* $P^H_{1,j}$ and $P^H_{n^l,j}$ are connected by the adjacent nodes in the bottom and top planes of the lowest and highest layers, respectively, as

$$\begin{aligned} P^H_{1,j} &= [p^b_{1,j}, p^b_{1,j+1}] \\ P^H_{n^l,j} &= [p^t_{n^l,j}, p^t_{n^l,j+1}] \end{aligned} \quad (j = 1, \dots, n^b) \tag{38}$$

- *saddle cables* are connected by the nodes in the top plane of layer k and the bottom plane of layer $k+1$ as

$$\begin{aligned} P^S_{k,2j-1} &= [p^t_{k,j}, p^b_{k+1,j}] \\ P^S_{k,2j} &= [p^b_{k+1,j}, p^t_{k,j+1}] \end{aligned} \tag{39}$$

where $j = 1, \dots, n^b$ and $k = 1, \dots, n^l - 1$.

The following relations have been used in Eqs. (38) and (39) for brevity.

$$p^b_{k,n^b+1} = p^b_{k,1}, \quad p^t_{k,n^b+1} = p^t_{k,1} \tag{40}$$

5.3.3 Vertical and Diagonal Cables

Connectivity of vertical and diagonal cables is not unique. For example, it may be noticed in Figs. 6.(a) and (b) that $p^t_{k,1}$ is connected to $p^t_{k,2}$ and $p^t_{k,3}$ by the vertical cables, respectively, leading to different topology. To illuminate this difference, we introduce the parameters c^v and c^d to define the connectivity of vertical and diagonal cables as follows:

- *vertical cable*: The connectivity of the vertical cables are defined by using an integer $c^v \in \{1, \dots, n^b\}$ as

$$P^V_{k,j} = [p^t_{k,j}, p^b_{k,j+c^v}] \tag{41}$$

where $j = 1, \dots, n^b$, $k = 1, \dots, n^l$, and $j + c^v = j + c^v - n^b$ if $j + c^v > n^b$.

- *diagonal cable*: The connectivity of the diagonal cables are defined by using an integer $c^d \in \{0, \dots, n^b - 1\}$ as

$$\begin{aligned} P^{D_b}_{k,j} &= [p^b_{k,j}, p^b_{k+1,j+c^d}] \\ P^{D_t}_{k,j} &= [p^t_{k,j}, p^t_{k+1,j+c^d}] \end{aligned} \tag{42}$$

  where $j = 1, \dots, n^b$, $k = 1, \dots, n^l - 1$, and $j + c^d = j + c^d - n^b$ if $j + c^d > n^b$.

From the connectivity of the members and nodes for a general n^l-layer tensegrity tower, with n^b struts in each layer, the numbers of struts m^b, horizontal cables m^h, vertical cables m^v, saddle cables m^s and diagonal cables m^d can be written as

$$\begin{aligned} &m^b = n^l n^b, && m^h = 2n^b, \\ &m^s = 2(n^l - 1)n^b, && m^v = n^l n^b \\ &m^d = 2(n^l - 1)n^b, && \end{aligned} \tag{43}$$

and the number m of all members of the structure is

$$m = 6n^l n^b - 2n^b \tag{44}$$

Following the definition of connectivity of each type of members and the numbering in Eq. (36), the connectivity matrix $\mathbf{C} \in \Re^{m \times n}$ can be easily constructed.

6. Numerical Examples

In this section, several numerical examples with different numbers of layers and struts in each layer are presented to investigate the validity of the proposed method and to demonstrate its capacity of systematically finding the configurations of tensegrity towers subjected to geometrical constraints.

In the following examples, the connectivities of the vertical and diagonal cables are fixed as $c^v = 1$ and $c^d = 0$ for simplicity.

6.1 Three-layer tensegrity tower

Consider a tensegrity tower as shown in Fig.7, which consists of three layers and four struts in each layer; i.e., $n^l = 3$ and $n^b = 4$. The structure is composed of 24 nodes and 64 members, including 12 struts, 8 horizontal cables, 12 vertical cables, 16 saddle cables and 16 diagonal cables. The members of each type in the same obit are classified into a group. Therefore, there exist 14 groups in total, and the members in the same group have the same force densities.

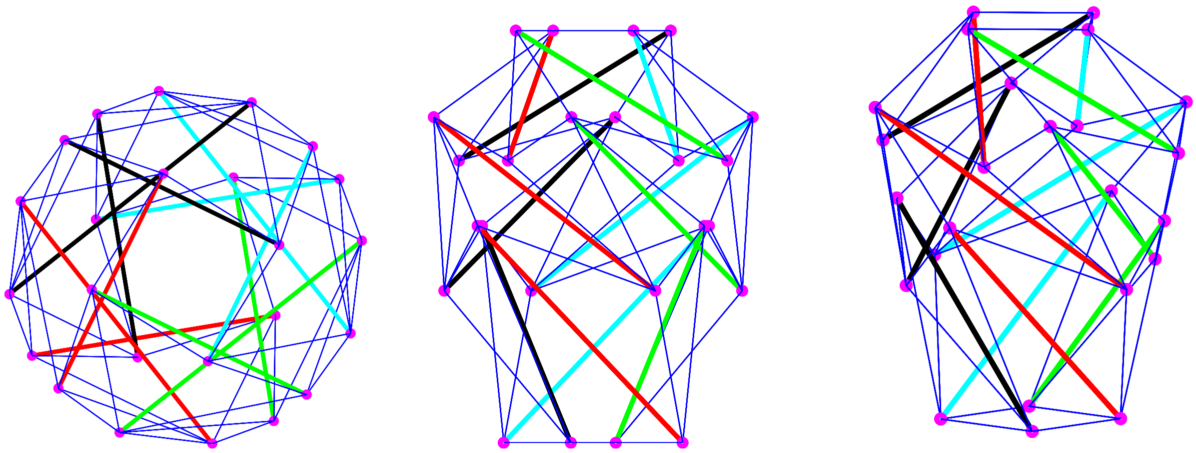
As an example, we assign the elevation of the structure as listed in Table 1. Note that the heights of every layer and the overlaps are not uniform. The total height H is 19.0.

To start Algorithm 1, the initial force densities of all struts and all cables are assigned as -1 and $+1$, respectively. Algorithm 1 runs 394 iterative steps for finding the feasible force densities as listed in Table 2, where q^{b_i}, q^{h_i}, q^{v_i}, q^{s_i} and q^{d_i} denote the force densities of the groups of struts, horizontal cables, vertical cables, saddle cables and diagonal cables, respectively.

The final equilibrium matrix $\hat{\mathbf{E}}$ has four zero eigenvalues and 20 positive eigenvalues, the minimum and maximum values of which are 0.1803 and 7.1592, respectively. Therefore, the structure satisfies the non-degeneracy condition and is super stable.

In the second design stage, there are up to four independent coordinates in xy-plane that can be arbitrarily specified by the designers, while the constraint on symmetry is considered.

Based on the algorithm of consecutively specifying the independent set of nodal coordinates described in [10], the xy-coordinates of the nodes $p^b_{1,1}$ and $p^t_{1,1}$ connected by the strut $P^B_{1,1}$ in the lowest layer are selected to be specified. Note that this is not the only independent set of coordinates. If the xy-coordinates of these two nodes are specified as $(10.0, 0)$ and $(2.5, 4.0)$, then the configuration of the structure is uniquely determined as shown



(a) Top view (b) Side view Perspective view

Fig. 7. A three-layer tensegrity tower with four struts in each layer.

Table 1
Elevation of the three-layer tensegrity tower.

H_1	H_2	H_3	h_1	h_2	h_3	H
10.0	8.0	6.0	0.0	3.0	2.0	19.0

Table 2
Feasible force densities of each group of members.

q^{b_1}	q^{b_2}	q^{b_3}	q^{h_1}	q^{h_2}	q^{v_1}	q^{v_2}	q^{v_3}
-1.1656	-1.1226	-1.2366	1.2758	1.2652		0.5718	
q^{s_1}	q^{s_2}		q^{d_1}	q^{d_2}	q^{d_3}	q^{d_4}	
1.4572	1.4547		0.8484	0.8262	0.5609	0.4190	

in Fig.7. It is easy to observed from the top view of the structure that the struts in the same layer are rotationally symmetric by the angle $\pi/2$.

If the same the nodal coordinates in xy-plane are specified to the strut $P^B_{2,1}$ in layer 2, the configuration is uniquely determined as shown in Fig.8. As can be easily observed, the new configuration of the structure becomes slightly slender compared with the configuration in Fig.7. Note that only the nodal coordinates in xy-plane have been changed in the second stage of the form-finding process. Therefore, Algorithm 1 need not be applied again to find the feasible force densities.

6.2 Ten-layer Tensegrity Tower

As an example of more complex structure, consider a ten-layer tensegrity tower as shown in Fig.9 with four struts in each layer; i.e., $n^l = 10$ and $n^b = 4$. The structure is composed of 80 nodes and 232 members. For simplicity, the heights and overlaps are uniformly assigned as $H_i = 10.0$ and $h_i = 2.0$ besides $h_1 = 0.0$, respectively. The total height is 82.0. Constraint on symmetry is also incorporated for this structure. The initial force densities of all struts and cables are given as -1 and $+1$, respectively.

After 511 iterative steps in Algorithm 1 for finding the feasible force density vector, four independent nodal coordinates in xy-plane need to be specified for this symmetric ten-layer tensegrity tower. If the xy-coordinates of the nodes $p^b_{2,1}$ and $p^t_{2,1}$ con-

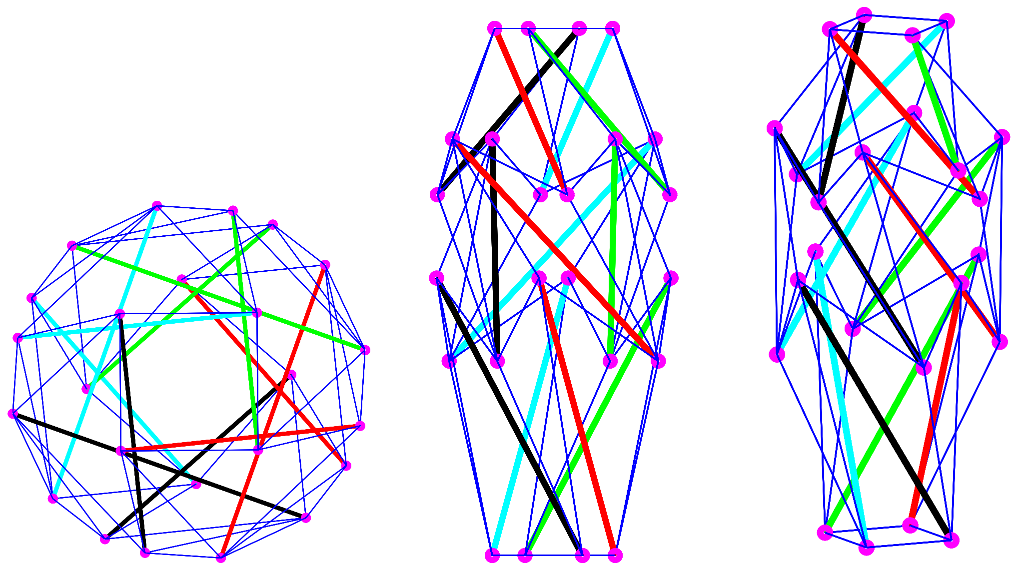


Fig. 8. New configuration of the symmetric three-layer tensegrity tower with the same force densities and coordinates in z-direction but different coordinates in xy-plane.

nected by the strut $P^B_{2,1}$ in layer 2 are specified as $(10.0, 0)$ and $(2.5, 4.0)$, the symmetric configuration of the structure is obtained as shown in Fig.9, where the top view and the side view have been drawn in different scales. The structure is super stable, because the equilibrium matrix is positive semi-definite with four zero eigenvalues. So the necessary condition for non-degeneracy is satisfied as well.

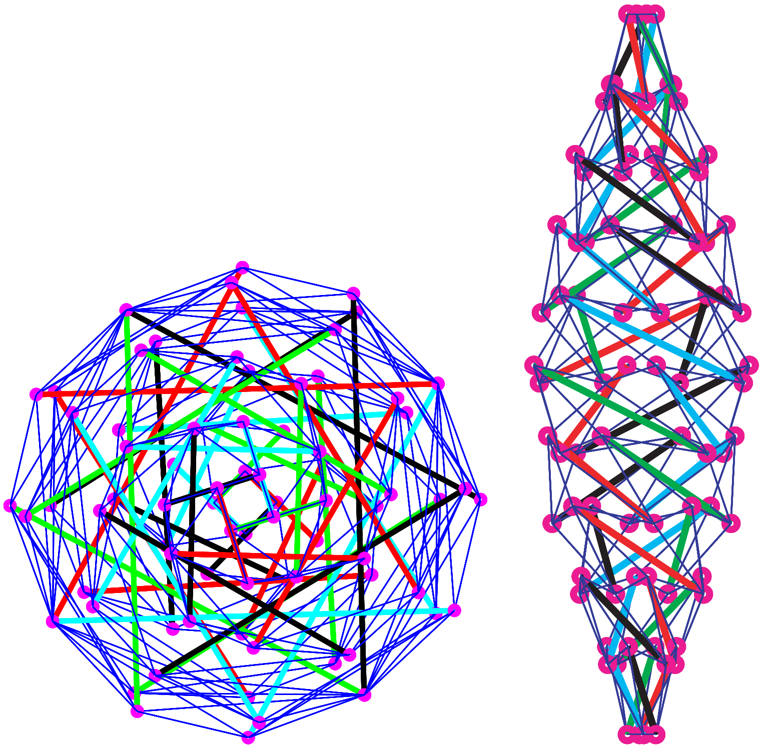
By modifying the values of the initial force densities and the independent nodal coordinates, more new interesting configurations can be systematically found. Hence, it can be concluded that the proposed method is applicable to a tensegrity tower with any number of layers ($n^l \geq 1$) and any number of struts in each layers ($n^b \geq 3$), although other examples of more complex structures are not presented.

7. Discussions and Conclusions

Two geometrical constraints, symmetry and elevation, of a tensegrity structure are incorporated into the adaptive force density method developed by the authors. The constraints are formulated in linear forms with respect to force densities and nodal coordinates, that are incorporated in the two design stages of the presented method The linear constraints with respect to the force densities are used in the first design stage to find the feasible force densities that satisfy the non-degeneracy condition. The constraints with respect to the nodal coordinates are combined with the self-equilibrium equations to uniquely determine the configuration in the second stage by assigning the independent nodal coordinates.

In the form-finding process of a tensegrity structure subjected to geometrical constraints, the following parameters need to be specified:

(1) topology;
(2) geometrical constraints;
(3) an initial set of force densities;
(4) an independent set of nodal coordinates.



(a) Top view (b) Side view

Fig. 9. A ten-layer tensegrity tower with four struts in each layer.

The tensegrity towers are used as numerical examples to illustrate the capability of finding appropriate configuration subjected to the geometrical constraints. Moreover, based on the detailed description of the geometry of the tensegrity towers, designers can avoid tedious job of modelling the topology of complicated tensegrity structure, and can concentrate on design aspect according to their preferences.

Besides the tensegrity towers, the proposed method can also be applied to other types of tensegrity structures subjected to geometrical constraints.

However, although the designers can have direct and exact controls over some geometrical properties, it is unlikely to control all aspects of a tensegrity structure. For example, it is not easy to exactly assign the lengths of all members. This is also considered as one of the common shortcomings of the family of force density method.

Acknowledgement:
The first author acknowledges support from The Kyoto University Foundation and thanks Dr. Simon D. Guest for the helpful discussions.

REFERENCES

[1] Fuller, R.B., 1975. Synergetics, Explorations in the Geometry of Thinking. Collier Macmillan, London, UK.

[2] Ohsaki, M. and Zhang, J.Y., Stability investigation of prestressed pin-jointed structures. submitted for pulication.

[3] Linkwitz, K. and Schek, H.-J., 1971. Einige Bemerkungen zur Berechnung von vorgespannten Seilnetzkonstruktionen. Ingenieur-Archiv. 40, 145–158.

[4] Vassart, N. and Motro, R., 1999. Multiparametered formfinding method: application to tensegrity systems. Int. J. Space Struct., 14(2), 147–154.

[5] Zhang, J.Y. and Ohsaki, M., in press. Adaptive force density method for form-finding problem of tensegrity structures. Int. J. Solids Struct.

[6] Harary, F., 1969. Graph Theory. Addison-Wesley, Reading, MA.

[7] Connelly, R. and Terrell, M., 1995. Globally rigid symmetric tensegrities. Structural Topology, 21, 59–78.

[8] Bishop, D.M., 1973. Group Theory and Chemistry. Calarendon Press, Oxford.

[9] Lay, D.C., 1996. Linear Algebra and Its Applications, 2nd edition. Reading, MA: Addison-Wesley.

[10] Zhang, J.Y., Ohsaki, M. and Kanno, Y., 2006. A direct approach to design of geometry and forces of tensegrity systems. Int. J. Solids Struct., 43, 2260–2278.

[11] Sultan, C., Corless, M. and Skelton R.E., 2002. Symmetrical reconfiguration of tensegrity structures. Int. J. Solids Struct., 39, 2215–2234.

[12] Zhang, J.Y., Guest, S.D. and Ohsaki, M., 2006. Symmetric Prismatic Tensegrity Structures: Part I. Configuration and Stability. in prepraration.