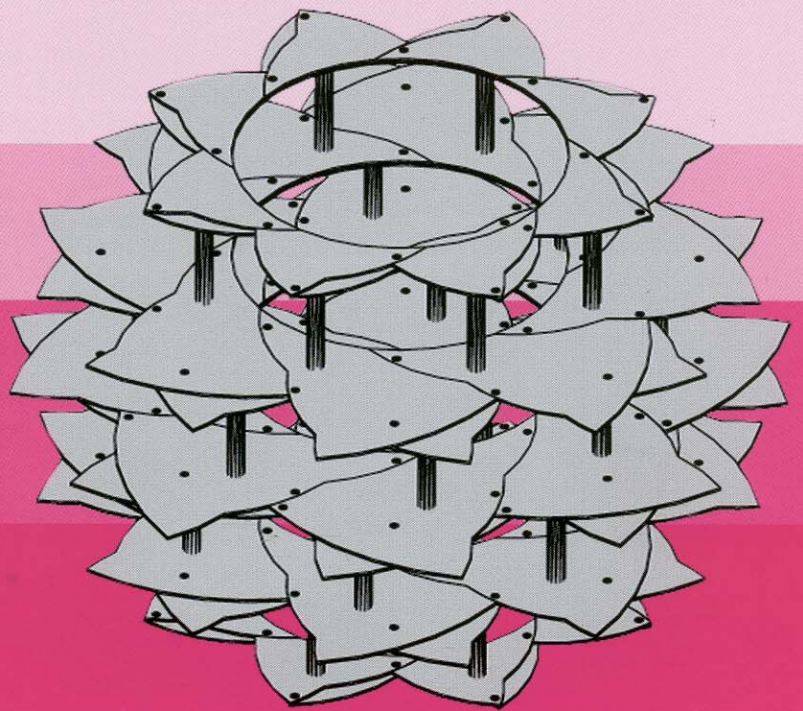


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Prof. D. h-C Eng .E. TORROJA, founder



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*COVER: Figure from F. Jensen and S. Pellegrino paper*

# FORM-FINDING OF SELF-STRESSED STRUCTURES BY AN EXTENDED FORCE DENSITY METHOD

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**Editor's Note:** This first author of this paper is one of four winners of the 2005 Hangai Prize, awarded for outstanding papers that are submitted for presentation and publication at the annual IASS Symposium by younger members of the association (under 30 years old). It is republished here with the permission of the editors of the proceedings of the IASS 2005 Symposium on Shell and Spatial Structures – Theory, Technique, Valuation, Maintenance, held in September 2005 in Bucharest, Romania.

## SUMMARY

*A numerical method for the form-finding problem of self-stressed structures is presented as an extension of the basic idea and formulations of the force density method, which is initially developed for cable nets. Singular value decomposition of the equilibrium matrix with respect to nodal coordinates is utilized to find the feasible set of force densities for satisfying the non-degeneracy condition of the structure. A unique configuration of the structure can be obtained by specifying an independent set of nodal coordinates. The proposed method can have some, although not exact or direct, controls over the geometrical and mechanical properties of the structure. Its strong ability of searching new configurations and the good convergence properties are demonstrated by several numerical examples.*

**Keywords:** self-stressed structure, form-finding, force density method, singular value decomposition.

## 1. INTRODUCTION

The self-stressed structure considered in this study belongs to the family of tensegrity systems classified by Motro [1]. It is composed of two distinct kinds of members: cables, which transmit only tensile forces, and struts, which transmit only compressional forces. Furthermore, it is free-standing, where no fixed node (support) exists.

Self-stressed structure is usually kinematically indeterminate in the unstressed state. In this case, the structure is also said to have mechanism(s) – some nodes may have displacements without extension of any member. Hence, the structure is usually unstable in the unstressed state. Moreover, self-stressed structure is always statically indeterminate so that there may exist self-stresses in its members. The mechanisms can be stiffened by the introduction of self-stresses so that the structure can maintain its stability in the prestressed state. This is considered to be one of the most distinct mechanical properties compared to any other kind of structures.

As the geometrical configuration and prestress distribution of a self-stressed structure are highly interdependent, the appropriate configuration and self-stresses have to be determined simultaneously by the process of *form-finding*.

The force density method [2], originally developed for the form-finding problem of cable nets, is very effective and straightforward, since only linear equations need to be solved. Based on the fact that almost the same assumptions have been adopted in the form-finding process of self-stressed structures and cable nets, a number of methods, which are extended from the basic idea and formulations of the force density method, have been proposed for the problem of self-stressed structures. Among these methods, the analytical technique presented in Ref.[3] is regarded as a very excellent approach for finding new configurations [4]. In this approach, force densities are represented and calculated in symbolic forms; Gaussian elimination is applied to the equilibrium matrix to find the necessary condition of the force densities that satisfies the non-degeneracy condition of the equilibrium matrix.

However, the equilibrium matrix in this analytical method has to be analyzed in symbolic forms, which is thought to be not effective enough for a relatively complicated structure, e.g. a structure with many nodes and members. So the objective of this study is to present a numerical method that finds the feasible set of force densities, based on Singular Value Decomposition (SVD) of the equilibrium matrix.

Following this introduction, the paper is organized as follows: In Section 2, the equilibrium matrix is formulated based on the basic idea of the force density method; non-degeneracy condition for the self-stressed structures is briefly described; approximate solution of the force density vector is obtained by using the equilibrium matrix and the given linear constraints on the force densities. In Section 3, the feasible set of force densities is found by applying SVD to the equilibrium matrix, and the proposed method is summarized. In order to illustrate the convergence and ability of searching new configurations of the proposed method, two- and three-stage self-stressed structures are investigated extensively in Section 4. Section 5 gives some discussions and concludes the study.

## 2. FORMULATIONS

### 2.1 Equilibrium Analysis

In order to simplify the form-finding problem of self-stressed structures, the following assumptions are introduced:

- topology of the structure is known;
- members are connected by pin joints;
- no external load is applied and the self-weight is neglected;
- buckling of the struts is not considered;
- the structure is free-standing.

From the assumptions (a)-(c), which are usually adopted for cable nets, we may have the following properties:

- configuration of the structure can be described in terms of the nodal coordinates;
- each member transmits only axial force, either in compression or in tension;
- the structure is in a state of self-equilibrium.

Consider a self-stressed structure with  $m$  members and  $n$  nodes. The connectivity of members and nodes, called *topology* of the structure, can be described by an incidence matrix  $\mathbf{C} \in \mathfrak{R}^{m \times n}$ : if member  $k$  is connected by nodes  $i$  and  $j$  ( $i < j$ ), then the  $i$ th and  $j$ th elements of the  $k$ th row of  $\mathbf{C}$  are 1 and -1, respectively, while the others are 0, which can be written as

$$\mathbf{C}_{(k,d)} = \begin{cases} 1 & \text{for } d=i \\ -1 & \text{for } d=j \\ 0 & \text{for other cases} \end{cases} \quad (1)$$

The force density  $q_k$  of member  $k$  is defined as the ratio of its axial member force  $s_k$  to the length  $l_k$ ; i.e.  $q_k = s_k/l_k$ . Positive and negative values of the force density  $q_k$  indicate that member  $k$  is in the states of tension and compression, respectively. The force density vector  $\mathbf{q} \in \mathfrak{R}^m$  is denoted as  $\mathbf{q} = (q_1, q_2, \dots, q_m)^T$ . Without any support, self-equilibrium equations of the structure in each direction with respect to the nodal coordinate vectors  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$  ( $\in \mathfrak{R}^n$ ) can be written as

$$\begin{aligned} \mathbf{E}\mathbf{x} &= \mathbf{0} \\ \mathbf{E}\mathbf{y} &= \mathbf{0} \\ \mathbf{E}\mathbf{z} &= \mathbf{0} \end{aligned} \quad (2)$$

Since the equilibrium of the structure is formulated by the matrix  $\mathbf{E} \in \mathfrak{R}^{n \times n}$  in Eq. (2),  $\mathbf{E}$  is called *equilibrium matrix* in this study, and it can be calculated by

$$\mathbf{E} = \mathbf{C}^T \mathbf{Q} \mathbf{C} \quad (3)$$

where the force density matrix  $\mathbf{Q}$  is the diagonal matrix of  $\mathbf{q}$ ; i.e.  $\mathbf{Q} = \text{diag}(\mathbf{q})$ . Note that  $\mathbf{E}$  is always square and symmetric.

It is obvious that the self-equilibrium equations are linear with respect to the nodal coordinates if the force density vector  $\mathbf{q}$  is given. In the case of a typical cable net, its configuration can be uniquely determined by directly solving the self-equilibrium equations, where  $\mathbf{E}$  is full-rank and right-hand sides of the equilibrium equations are non-trivial for a typical cable net since it is suspended by supports. This is the basic idea of the force density method for cable nets.

However, in the case of self-stressed structure,  $\mathbf{E}$  is always *singular* with rank deficiency of at least 1, because the sum of elements of any column or row of  $\mathbf{E}$  is equal to 0. This can be easily verified from the fact that the structure does not have supports, and the values of the force densities may be negative or positive. Hence, configuration of the structure, which is described by  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{z}$ , cannot be uniquely determined by solving Eq. (2). Define the rank deficiency  $h$  of  $\mathbf{E}$  as

$$h = n - \text{rank}(\mathbf{E}) \quad (4)$$

From Eq. (2), there are up to  $h$  independent nodal coordinates in  $x$ -,  $y$ - and  $z$ -directions, respectively, that can be arbitrarily specified.

Geometrically, three and four independent nodes can construct two- and three-dimensional structures, respectively. Therefore, the required rank deficiency  $h^*$  of the equilibrium matrix  $\mathbf{E}$  has to satisfy  $h^* \geq 4$ , which is the necessary condition for a three-dimensional non-degenerate self-stressed structure. In the following, we will mainly concentrate on the case of  $h^* = 4$ .

## 2.2 Formulation of the Force Density Vector

Instead of using Eq. (3), the equilibrium matrix  $\mathbf{E}$  can be constructed directly from the force densities. Let  $J$  denote the set of members connected to node  $i$ . The  $(i, j)$  component  $\mathbf{E}_{(i,j)}$  of  $\mathbf{E}$  can be given as

$$\mathbf{E}_{(i,j)} = \begin{cases} \sum_{k \in J} q_k & \text{for } i = j \\ -q_k & \text{if } i \text{ and } j \text{ are connected by member } k \\ 0 & \text{for other cases} \end{cases} \quad (5)$$

From the definition of the equilibrium matrix  $\mathbf{E}$  in Eq. (5), the  $i$ th column  $\mathbf{E}_i$  of  $\mathbf{E}$  can be written in terms of the force density vector  $\mathbf{q}$  by the matrix  $\mathbf{B}^i \in \mathfrak{R}^{n \times m}$  as

$$\mathbf{B}^i \mathbf{q} = \mathbf{E}_i \quad (6)$$

where the component  $\mathbf{B}_{(j,k)}^i$  of  $\mathbf{B}^i$  is defined as

$$\mathbf{B}_{(j,k)}^i = \begin{cases} 1 & \text{for } i = j \text{ and } k \in J \\ -1 & \text{if } i \text{ and } j \text{ are connected by member } k \\ 0 & \text{for other cases} \end{cases} \quad (7)$$

Hence, the force density vector  $\mathbf{q}$  can be expressed in terms of components of  $\mathbf{E}$  as follows by letting  $\mathbf{B}^T = (\mathbf{B}^{1T}, \dots, \mathbf{B}^{iT}, \dots, \mathbf{B}^{nT})$  and  $\mathbf{g}^T = (\mathbf{E}_1^T, \dots, \mathbf{E}_i^T, \dots, \mathbf{E}_n^T)$ :

$$\mathbf{B} \mathbf{q} = \mathbf{g} \quad (8)$$

Linear constraints, such as direct assignment of the values, or relations due to symmetry conditions, on some specific force densities can be formulated as

$$\mathbf{B}^e \mathbf{q} = \mathbf{q}^e \quad (9)$$

where  $\mathbf{B}^e$  and  $\mathbf{q}^e$  denote the constraint matrix and the vector having the expected values, respectively. By letting  $\tilde{\mathbf{B}}^T = (\mathbf{B}^T, \mathbf{B}^{eT})$  and  $\tilde{\mathbf{g}}^T = (\mathbf{g}^T, \mathbf{q}^{eT})$ , we may obtain

$$\tilde{\mathbf{B}} \mathbf{q} = \tilde{\mathbf{g}} \quad (10)$$

Since matrix  $\tilde{\mathbf{B}}$  is full-rank, known by its definition for a self-stressed structure, the least square solution of the force density vector  $\mathbf{q}$  can be written as

$$\mathbf{q} = \tilde{\mathbf{B}}^- \tilde{\mathbf{g}} \quad (11)$$

where  $\tilde{\mathbf{B}}^-$  denotes the generalized inverse of  $\tilde{\mathbf{B}}$ .

## 3. EXTENDED FORCE DENSITY METHOD

### 3.1 Singular Value Decomposition

Singular Value Decomposition (SVD) is a very powerful technique which can deal with a set of linear equations or matrices that are either singular or numerically very close to singular [5]. By implementation of SVD, any matrix  $\mathbf{E} \in \mathfrak{R}^{n \times n}$  can be decomposed and written as the product of a column-orthogonal matrix  $\mathbf{U} \in \mathfrak{R}^{n \times n}$ , a diagonal matrix  $\mathbf{\Lambda} \in \mathfrak{R}^{n \times n}$  with non-negative elements and the transpose of a column-orthogonal matrix  $\mathbf{V} \in \mathfrak{R}^{n \times n}$ :

$$\mathbf{E} = \mathbf{U} \mathbf{\Lambda} \mathbf{V}^T \quad (12)$$

where

$$\begin{aligned} \mathbf{U}^T \mathbf{U} &= \mathbf{I} \\ \mathbf{V}^T \mathbf{V} &= \mathbf{I} \end{aligned} \quad (13)$$

and  $\mathbf{I} \in \mathfrak{R}^{n \times n}$  denotes the identity matrix. The diagonal elements of  $\mathbf{\Lambda}$  are called *singular values*. If the rank of  $\mathbf{E}$  is  $r$ , then it has  $r$  positive and  $n - r$  additional zero singular values. The positive singular values  $\lambda_1, \lambda_2, \dots, \lambda_r$  are numbered in non-increasing order as

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0 \quad (14)$$

Consider a case where  $n - r < h^*$ , which is a general situation for an arbitrary chosen set of force densities. Since setting the smallest positive singular values of  $\mathbf{E}$  to zero may lead to minimal modification of the equilibrium matrix, we may set the  $h^*$  smallest singular values to zero in order to make the equilibrium matrix  $\mathbf{E}$  have the required rank deficiency  $h^*$ :

$$\lambda_{n-h^*+1} = \lambda_{n-h^*+2} = \dots = \lambda_n = 0 \quad (15)$$

Let  $\bar{\mathbf{\Lambda}}$  denote the updated  $\mathbf{\Lambda}$  so that it has  $h^*$  null diagonal terms. The updated version  $\bar{\mathbf{E}}$  of the equilibrium matrix that has the required rank deficiency  $h^*$  can be derived as follows:

$$\bar{\mathbf{E}} = \mathbf{U} \bar{\mathbf{\Lambda}} \mathbf{V}^T \quad (16)$$

A new force density vector  $\bar{\mathbf{q}}$  can then be achieved by Eq. (11) with the new  $\bar{\mathbf{E}}$  that is incorporated to compute  $\tilde{\mathbf{g}}$ .

### 3.2 Form-finding Process

Since Eq. (11) is an approximate solution of the force density vector  $\mathbf{q}$ , the equilibrium matrix calculated from  $\mathbf{q}$  using Eq. (3) may not have the required rank deficiency exactly. Therefore, the following iterative computation is necessary for finding the feasible force densities.

#### Algorithm 1: Feasible Force Densities

**Step 0:** Specify an initial force density vector  $\mathbf{q}^0$  to obtain  $\mathbf{E}^0$  by Eq. (3). Construct  $\tilde{\mathbf{B}}$  and  $\tilde{\mathbf{g}}^0$  by combining Eqs. (8) and (9). Set  $i := 0$ .

**Step 1:** Set the  $h^*$  smallest singular values of  $\mathbf{E}^i$  to zero to reconstruct  $\bar{\mathbf{E}}^i$  by Eq. (16).

**Step 2:** Obtain  $\tilde{\mathbf{g}}^{i+1}$ , calculate  $\mathbf{q}^{i+1}$  from Eq. (11) and update  $\mathbf{E}^{i+1}$  by Eq. (3).

**Step 3:** Check the rank of  $\mathbf{E}^{i+1}$ ; i.e. if it satisfies  $n - \text{rank}(\mathbf{E}^{i+1}) = h^*$  exactly, then let  $\hat{\mathbf{q}} = \mathbf{q}^{i+1}$  and  $\hat{\mathbf{E}} = \mathbf{E}^{i+1}$ , and terminate the algorithm; otherwise, set  $i \leftarrow i + 1$  and return to Step 1.

This way, we can achieve the feasible force density vector  $\hat{\mathbf{q}}$ , the corresponding equilibrium matrix  $\hat{\mathbf{E}}$  of which has the required rank deficiency  $h^*$ .

To obtain the configuration of the structure, the nodal coordinates should be determined uniquely. Compute  $\mathbf{H} \in \mathfrak{R}^{3n \times 3n}$  by the tensor product of the identity matrix  $\mathbf{I} \in \mathfrak{R}^{3 \times 3}$  and  $\hat{\mathbf{E}}$  as

$$\mathbf{H} = \mathbf{I} \hat{\mathbf{E}} \quad (17)$$

The self-equilibrium equations of the structure in each direction can then be combined as follows:

$$\mathbf{H}\mathbf{X} = \mathbf{0} \quad (18)$$

where  $\mathbf{X}^T = (\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T)$ . The solution of Eq. (18) can be written as

$$\mathbf{X} = \mathbf{P}\boldsymbol{\beta} \quad (19)$$

where  $\mathbf{P} \in \mathfrak{R}^{3n \times 3h^*}$  satisfies  $\mathbf{H}\mathbf{P} = \mathbf{0}$ , and  $\boldsymbol{\beta} \in \mathfrak{R}^{3h^*}$  is the coefficient vector. By specifying an independent set of nodal coordinates  $\bar{\mathbf{X}}$ , the configuration of the structure can then be uniquely determined by

$$\mathbf{X} = \mathbf{P}\bar{\mathbf{P}}^{-1}\bar{\mathbf{X}} \quad (20)$$

where  $\bar{\mathbf{P}}$  is defined by the rows of  $\mathbf{P}$  corresponding to the specified  $\bar{\mathbf{X}}$ . An algorithm that enables designers to specify an independent set of nodal coordinates consecutively can be found in Ref.[6].

The design procedure of the proposed method can be summarized as follows:

#### Algorithm 2: Design Procedure

**Step 1:** Formulate the linear constraints on force densities, and specify an initial set of force densities.

- Step 2:** Obtain the feasible set of force densities by implementation of Algorithm 1.
- Step 3:** Specify an independent set of nodal coordinates to obtain a unique configuration of the structure.

**4. NUMERICAL EXAMPLES**

The proposed method is implemented by MATLAB Ver. 6.5.1 [7]. Convergence of Algorithm 1 and ability of searching new configurations of the proposed approach are investigated in this section. In the following examples, the singular value is regarded as zero if its absolute value is smaller than  $10^{-16}$ ; thick and thin lines in the figures denote struts and cables, respectively; and no linear constraint on the force densities is considered.

First, a two-stage self-stressed structure as shown in Fig. 1 is considered. It is composed of ( $n=$ )12 nodes and ( $m=$ )30 members, including six struts and 24 cables. For simplicity, struts of the structure are divided into two groups (1) lower stage and (2) upper stage; and its cables are divided into: (3) top and bottom bases, (4) saddle, (5) vertical and (6) diagonal as described in Ref.[8].

**Example 1:**

If the initial set of force densities for the six groups is specified as ‘Initial’, then the feasible set of force

densities is obtained as ‘Final’ listed in Table 1 by implementation of Algorithm 1. Note that all members in the same group have the same values of force densities in this example.

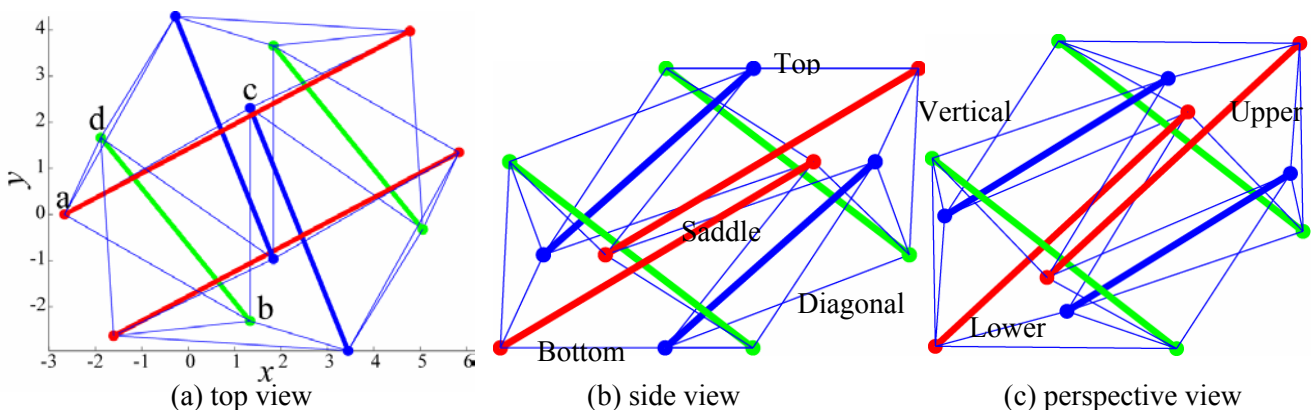
If we specify the coordinates of the three independent nodes ‘a’, ‘b’ and ‘c’ defined in Fig. 1(a) to locate the bottom base on the xy-plane, and node ‘d’ in the lower stage as listed in Table 2, we can then achieve the configuration of the structure as shown in Fig. 1. It can be observed in this example that the members of each group are rotationally symmetric, although it may not be always the case, e.g. as demonstrated in Example 4.

*Table 1. Force densities of Example 1*

Group	1	2	3	4	5	6
Initial	- 1.0	- 1.0	1.0	1.0	1.0	1.0
Final	- 1.56	- 1.56	1.04	1.04	1.04	1.04

*Table 2. Specified nodal coordinates in Example 1*

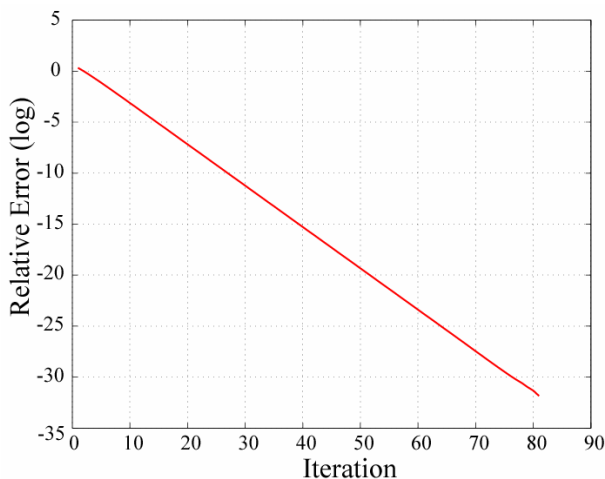
Node	a	b	c	d
x	- 2.6667	1.3333	1.3334	- 1.8867
y	0.0	- 2.3094	2.3094	1.6666
z	0.0	0.0	0.0	3.3333



**Figure 1.** Example 1: two-stage self-stressed structure.

In Example 1, Algorithm 1 terminates after 81 iterations to achieve the feasible set of force densities, where the corresponding equilibrium matrix has the required rank deficiency of four. The relative error of the force density vector is defined as the Euclidean norm  $\| \mathbf{q}^i - \hat{\mathbf{q}} \|$  of the difference of the force density vector  $\mathbf{q}^i$  at the  $i$ th iteration to the final force density vector  $\hat{\mathbf{q}}$ . The relative error in logarithmic scale is plotted in Fig. 2, from which a very good and rapid convergence of Algorithm 1 can be seen.

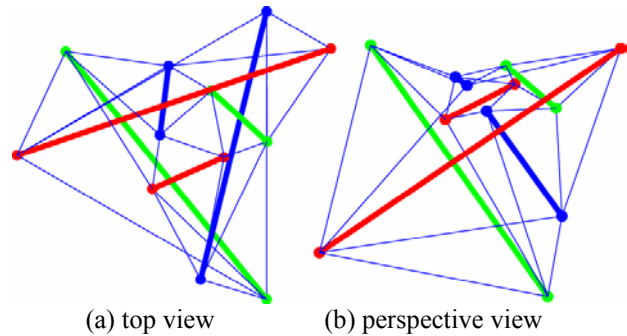
It is clear from the design procedure that we can obtain new configurations by simply modifying the values of the initial force densities at the first step, or the independent nodal coordinates at the last step of the form-finding process. The following two examples show how to search for new configurations of the structures with the same topology.



**Figure 2.** Convergence of Algorithm 1 applied in Example 1 for finding the feasible force densities

### Example 2:

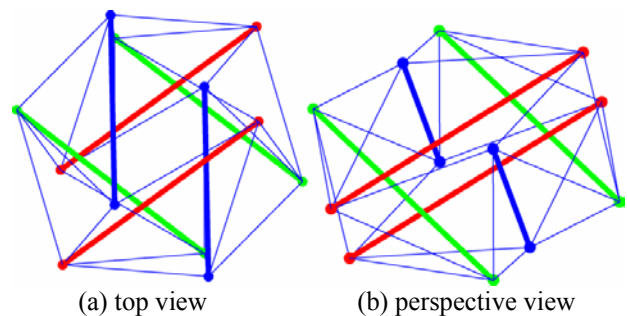
If the initial values of the force densities for the six groups are specified as  $\{-1.5, -0.7, 1.0, 2.0, 1.0, 1.0\}$  as Example 2, and the same nodal coordinates are specified as in Example 1, we may obtain the structure as shown in Fig. 3. It may be interesting to observe that the “upper” struts become shorter than the “lower” ones, and the “upper” stage is completely surrounded by the “lower” stage.



**Figure 3.** Example 2: two-stage self-stressed structure

### Example 3:

Besides changing the initial force densities at the first step of the form-finding process, we can modify the nodal coordinates at the last step as well in order to search for new configurations. If we modify the y-coordinate of node ‘a’ from  $-1.8667$  in Example 1 to  $-3.8667$ , and do not change any other specified value as Example 3, configuration of the structure is achieved as shown in Fig. 4.



**Figure 4.** Example 3: two-stage self-stressed structure

### Example 4:

As have been pointed out previously that rank deficiency of the equilibrium matrix has to satisfy  $h^* \geq 4$  for a non-degenerate three-dimensional self-stressed structure, and three numerical examples in which  $h^* = 4$  have been shown above. In some cases, where more controls over the coordinates of some nodes are desired, we can also make the equilibrium matrix  $\mathbf{E}$  have rank deficiency of more than four, so that more nodal coordinates can be specified according to the preference of designers.

By using the same initial force densities in Example 1 and making  $h^*=5$  for the final equilibrium matrix, we obtain the feasible set of force densities, where the force densities of each member in the same group are not identical any more; e.g. the force densities of the three struts in the lower stage are - 1.2471, - 1.2559 and - 1.5787. If we specify the coordinates of node 'e' defined in Fig. 5(a) as (4.0, - 3.0, 6.0), and give the same coordinates to the other four nodes as in Example 1, we can then obtain the configuration of the structure as shown in Fig. 5. It can be easily observed from the figure that the structure is asymmetric, even though we have specified an initial symmetric set of force densities for the members.

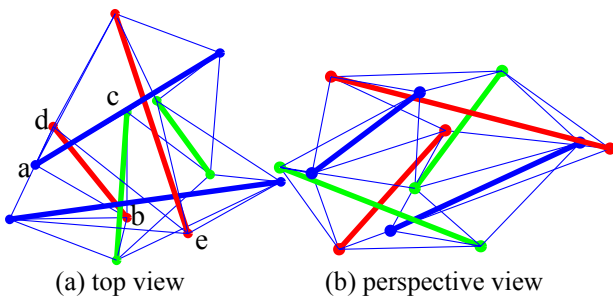


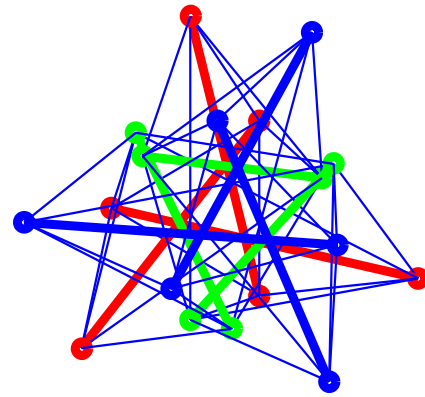
Figure 5. Example 4: two-stage self-stressed structure

**Example 5:**

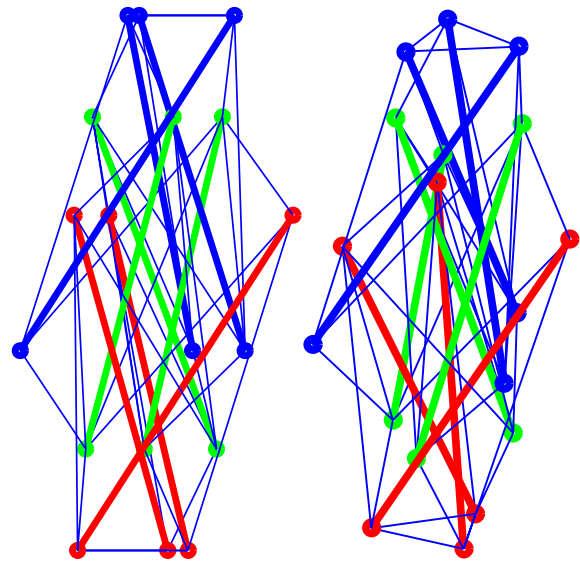
For a structure with more complicated topology than the two-stage self-stressed structure, a three-stage structure as shown in Fig. 6, consisting of ( $m=$ )48 members including 9 struts and 39 cables and ( $n=$ )18 nodes, is investigated. Struts of the structure are classified into two groups: (1) the six struts in the upper and lower stages and (2) the three struts in the center stage. The cables are classified into the same groups as the two-stage structure described previously.

Let the initial set of force densities for the six groups have values of  $\{- 0.8, - 1.0, 0.8, 0.3, 0.7, 0.5\}$  to obtain the feasible set of force densities by implementation of Algorithm 1, where the equilibrium matrix is made to have rank deficiency of four. By specifying the nodal coordinates of the bottom base and a lower node in the center stage as  $\{(- 2.6667, 0.0, 0.0), (1.3333, - 2.3094, 0.0), (1.3334, 2.3094, 0.0)\}$  and  $(- 2.0, 2.0, 4.0)$ , respectively, the unique configuration of the structure is obtained as shown in Fig. 6.

Stability of a structure can be checked by investigation of its tangent stiffness matrix where the rigid-body motions have been constrained; i.e., if the eigenvalues of the tangent stiffness matrix excluding the rigid-body motions are all positive, then the structure is stable; otherwise, it is unstable. All structures obtained in the numerical examples in this section have been confirmed to be stable.



(a) top view



(b) side view

(c) perspective view

Figure 6. Example 5: three-stage self-stressed structure

**5. DISCUSSIONS AND CONCLUSIONS**

A numerical method for the form-finding problem of self-stressed structures has been proposed as an extension of the basic idea and formulations of the force density method, which is initially developed for cable nets. By applying singular value decomposition to the equilibrium matrix and setting a specific number of smallest singular values to zero, the presented algorithm can find a feasible set

of force densities, which ensures that the corresponding equilibrium matrix has the required rank deficiency. An independent set of nodal coordinates, which can be specified according to the preference of designers, is utilized to determine a unique geometrical configuration of the structure, which is described in terms of nodal coordinates.

From the formulations and the given numerical examples, we may have the following conclusions and discussions on the proposed approach:

1. The algorithm, which finds the feasible set of force densities, has a good convergence property as demonstrated in Example 1.
2. To find a unique configuration, only topology, initial force densities given at the first step and an independent set of nodal coordinates specified at the last step are required.
3. The proposed approach has also a very strong ability of systematically searching for new geometrical configurations by changing the values of the initial force densities and the nodal coordinates to be specified.
4. However, it may not have exact or direct control over the geometrical and mechanical properties of the structure, since the parameters in the family of force density methods are neither forces nor lengths but the force-to-length ratios. As mentioned in Ref.[3], less symmetric or even asymmetric configurations may be found for a given symmetric set of force densities, because the member lengths cannot be described explicitly and linearly in the formulation.

How to formulate the symmetric and some other geometrical constraints of the self-stressed structure in a linear form of the force density vector, in order to have more direct control over the geometrical properties of the structure, may be the future study.

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